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Optimal experimentation and the perturbation method in the neighborhood of the augmented linear regulator problem

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Abstract

The perturbation method is used to approximate optimal experimentation problems. The approximation is in the neighborhood of the linear regulator (LR) problem. The first order perturbation of the optimal decision under experimentation is a combination of the LR solution and a term that captures the impact of the uncertainty on the agent's value function. An algorithm is developed in a companion paper to quickly implement this procedure on the computer. As a result, the impact of optimal experimentation on an agent's decisions can be quantified and estimated for a large class of problems encountered in economics.

JEL classification: C44; C60; D81; D82

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1. Introduction

Economic agents usually find themselves in circumstances in which they have to maximize their welfare while at the same time they must learn about fundamental

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relations that influence their payoffs. Optimal experimentation arises when the actions of the agents impact their information set. In these circumstances the agents trade off optimal control with optimal experimentation. In general these problems are difficult to solve since the actions of the agents affect the distribution of payoffs. The microeconomic literature has tended to focus on fixed states and non-linear payoffs. Within this context researchers have characterized the tradeoffs between myopic behavior and the value of information. The macroeconomic literature is focused more on changing state variables with quadratic payoffs. This paper demonstrates how to use the perturbation method to approximate the solution to experimentation problems found in macroeconomics.

The presence of optimal experimentation leads to stochastic problems with no closed form solution. However, in the absence of optimal experimentation most macroeconomic problems are reduced to augmented linear regulator (LR) problems. In addition, most dynamic general equilibrium models of the economy are approximated with log linearization methods which turn the model into an LR problem. These LR problems have well defined solutions which involve iteration on a Ricatti equation. This opens up the possibility of using perturbation methods to approximate optimal experimentation problems in the neighborhood of the solution to LR problems. In this procedure a parameter is introduced in which the optimal experimentation problem reduces to the LR problem when the parameter is zero. Subsequently, a Taylor series expansion around the zero value of the parameter is calculated for the value function and optimal policy rule. In this paper a second order expansion of the value function and a first order expansion of the policy function is found. Thus, the dynamic general equilibrium models with optimal experimentation can be approximated with the perturbation method introduced here with the same degree of accuracy as log linearization methods.

The next section discusses the relation between this work and research on optimal experimentation. Section 3 summarizes the procedures to solve LR problems following Anderson et al. (1996). Section 4 develops the parameterization of the conditional variance–covariance matrix of the optimal experimentation problem so that this problem reduces to the Kalman filtering problem when the parameters are set equal to zero. Section 5 derives the formula for optimal conditions in the presence of a second order perturbation of the optimal experimentation problem. The stability properties of this solution are also examined. In Section 6 the analysis of Balvers and Cosimano (1990) is applied to a bank with some monopoly power that does not know the demand for loans or the supply of deposits. This example is a generalization of Balvers and Cosimano in that optimal experimentation for two separate relations is undertaken. In addition, the state vector includes eleven variables, including lagged dependent variables, which moves stochastically over time. The results of Balvers and Cosimano are quantified in that both the loan and deposits rates slowly adjust to changes in market conditions. The final section concludes the paper.

2. Related literature

Original work on the optimal experimentation for economic problems was carried out by MacRae (1972, 1975) and Prescott (1971, 1972). Kendrick (2002a, b) discusses

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the role of optimal experimentation in stochastic control problems. He also provides a comprehensive discussion of earlier developments for this problem. Easley and Kiefer (1988), Kiefer and Nyarko (1989), Balvers and Cosimano (1990), Aghion et al. (1991), Trefler (1993), Keller and Rady (1999), Bolton and Harris (1999), and Moscarini and Smith (2001) are a few of the papers which analyze experimentation problems in microeconomic settings. Balvers and Cosimano (1994), and Wieland (2000a, 2006), provide discussions of the literature on optimal experimentation in macroeconomic problems.

Wieland (2000a) shows how to use value and policy function iteration to examine the behavior of optimal experimentation problems.[1] He introduced the optimal experimentation problem into otherwise static optimization problem.[2] Within the context of these problems he was able to identify as much as a 52% increase in the agent's value from experimentation.

Extending this procedure to more general problems or using it for the estimation of experimentation effects is problematic. In the case of a continuous distribution in which the agent is learning about four state variables Wieland's computer program takes about a week to find a solution.[3] Wieland (2002) evaluates a more complex macroeconomic problem in which agents learn about nine state variables. However, computational time, about 60 h, limits the discussion to pairwise evaluation of the complete optimal experimentation problem. Subsequently, Wieland (2006) examines optimal learning about the Phillips curve when there is uncertainty about the natural rate and the persistence of inflation. In this work it takes about 72 h to solve a model with five state variables. This limits the application of this procedure to simple calibration exercises and precludes the estimation of models of optimal experimentation.[4] In addition, optimal experimenting about multiple parameters or more complex dynamic settings is prohibitive.

This paper provides an alternative procedure for approximating optimal experimentation problems which can handle more complex optimal experimentation problems, as well as the estimation of these problems. This alternative procedure is based on the perturbation method of Gaspar and Judd (1997), Judd (1998), and Jin and Judd (2002). This method may be applied when the solution to the problem can be represented by a Taylor series.[5] The perturbation method is useful when a more general problem reduces to a simpler problem under some well-defined circumstances. In addition, the simpler problem has a well-developed solution method. The perturbation method proceeds to introduce parameters such that the general problem reduces to the simpler problem when these parameters are zero. The more

[1] Beck and Wieland (2002) use the same technique when the state variable is dependent on one lag of the dependent variable.

[2] Balvers and Cosimano (1990, 1994) and Wieland (2000b) provide examples in which this approach may be applied.

[3] Over the years the increased capacity of computers has resulted in a significant reduction in this time.

[4] Wieland (2000b) provides a good example of a calibration exercise.

[5] Chen et al. (2005) show that solution to discrete time asset pricing problems can be represented as a Taylor series when the components of the problem can also be represented by a Taylor series. Such functions are called analytic.

general problem is then approximated by taking a Taylor series approximation around zero values for the perturbation parameters.

In most macroeconomic problems on optimal experimentation the objective is quadratic and the equations of motions are linear. It is also possible to introduce parameters which remove the optimal experimentation problem. For example in Wieland's (2000b) optimal monetary policy problem the optimal experimentation issue would not be present when the central bank knows the impact of interest rates on inflation. Consequently, the optimal experimentation may be removed by attaching a parameter to the error term for this slope coefficient and setting the parameter to zero.[6]

Without the optimal experimentation these problems fall into the general rubric of LR problem. Also it is traditional in simulations of dynamic general equilibrium models to use a second order approximation of the value function and a first order approximation of the equations of motions which turns these models into a LR problem. For example, one can use the computer algorithm *Dynare* to do these approximations. The procedures for solving the LR problems have been developed by Hansen and Sargent (1998), and Anderson et al. (1996). As a result, one can use the perturbation method to approximate the optimal decision of an agent in the presence of experimentation. In this paper the second order perturbation to the experimentation problem is found in the neighborhood of the LR solution. The optimal decision starts with the optimal decision found in Anderson et al. Appended to this decision is a term which captures the effect of the conditional variance–covariance matrix on the agents' optimal decisions. Thus, it is now feasible to examine the impact of optimal experimentation on any dynamic general equilibrium model.

This additional impact on the optimal decisions of the agent is akin to Ito's lemma in continuous time stochastic control problems.[7] The effect of optimal experimentation works through the second order effect of the variance–covariance matrix on the agent's marginal valuation of each state. This effect may be decomposed into two parts based on the chain rule. The variance–covariance matrix first changes how each state variable impacts the equation of motion for the state variables. This part occurs because this change influences the Kalman gain. The second part consist of the marginal impact of the state variables on the agent's valuation. The perturbation method develops a systematic way to measure these effect of optimal experimentation which is then added to the optimal decision from the LR problem.

There are several benefits to this procedure. First, it builds on a well-developed procedure for handling a large class of economic problems. This class of problems includes those found in the literature on optimal experimentation, as well as dynamic general equilibrium models. Second, the procedure can be implemented on a

[6]A similar set up could be used to apply the perturbation method to Balvers and Cosimano (1990, 1994).

[7]Keller and Rady (1999), Bolton and Harris (1999), and Moscarini and Smith (2001) provide examples of an optimal experimentation problem in continuous time. Bensoussan (1992) provides mathematical analysis of latent variables and stochastic control in continuous time.

computer in a timely manner.[8] Finally, by suitably modifying the estimation procedure of Anderson et al. it is now feasible to estimate optimal decision rules of agents in the presence of optimal experimentation. For example Sargent (1999) develops an optimal monetary control problem which fits into the Anderson et al. framework.[9] This optimal monetary control problem is a more general version of Wieland (2000b). Thus, it would be feasible to estimate an optimal central bank reaction function in the presence of optimal experimentation.

The main drawback of this procedure is that it assumes that the value function and optimal decisions are differentiable in the neighborhood of the LR solution. Otherwise a Taylor series is not well defined for the solution. There are at least three reasons found in the literature when the value or policy functions are not differentiable. First, Balvers and Cosimano (1993) develop an example in which the objective of the agent is convex. Earlier, Easley and Kiefer (1988) showed that the value function is convex in the conditional distribution of the shocks. As a result, the Bellman equation is convex. Thus, the optimal decision is a corner solution. Balvers and Cosimano show that the value function is not differentiable when the agents switches between corner solutions. This problem would not occur when the one period reward function is sufficiently concave relative to the convex effect of experimentation.[10] This condition may be checked in the perturbation method by examining whether a particular matrix is negative definite in the neighborhood of the LG solution.[11] Next, Santos (1994) shows that the policy function also fails to be differentiable when the dynamics are such that the system converges to a corner solution. This problem can be identified by examining the stability of the system of equations as in Section 5.

Finally, Keller and Rady (1999) and Wieland (2000a) show that the value and policy functions are not differentiable when the agent has incorrect limit beliefs. Following Kiefer and Nyarko (1989), Wieland identifies three properties of limit beliefs and optimal decisions. First beliefs must be self-reinforcing. Second, given the limit beliefs the reward function should be optimized. Third, if the control variable is held constant, the agent would learn the mean value of the latent variables. The possible solution to these conditions are incorrect when the expected values of the latent parameters do not converge to their true value. In the simulations Keller and Rady (1999), and Wieland (2000a) find that the value function and optimal decisions are not differentiable when the latent parameter corresponds to an incorrect limit belief. Thus, the procedure needs to avoid these possibilities.

[8]The computer program for this paper is tailored to the specific example. Subsequently, Cosimano and Gapen (2006) have developed a computer algorithm in Matlab which can be applied to any log-linearize dynamic general equilibrium model. Running this algorithm takes a few seconds on a standard PC.

[9]Wieland's (2002) model of monetary policy under uncertainty about the natural rate of unemployment and Woodford's (2002) model of imperfect common knowledge could also fit within this class of models.

[10]Keller and Rady (1999) analyze a similar problem when the agent's objective is concave. In this case corner solutions arise when the experimentation effect is dominant. However, in this continuous time case the value function is differentiable.

[11]This matrix is effectively an approximation of the Hessian matrix for the Bellman equation.

In Cosimano and Gapen (2006) the model of Beck and Wieland (2002) is simulated using the computer algorithm which implements the perturbation method developed here. Kendrick (2006) is also working on simulating the Beck and Wieland model to identify the cost and benefits of his procedure, developed in Kendrick (2002a), relative to the other procedures. Cosimano and Gapen find that the perturbation method and dynamic programming method provide the same solution when there is no discontinuity in the optimal control. However, the perturbation method cannot pick up the circumstance in the Beck and Wieland model associated with incorrect limiting beliefs. As a result, the dynamic programming methods of Wieland (2000a) must be used to help identify regions of incorrect limiting beliefs. Under such circumstances the use of the perturbation method is inappropriate.

Using the perturbation method in the neighborhood of the LR problem to approximate optimal experimentation problems fails to capture the convergence to incorrect limit beliefs and the convergence to corner solutions. The reason is that perturbation methods provides an approximations within the neighborhood of the LR problem so that it is only a local solution. Starting with Marcet and Sargent (1989) the literature on optimal learning has developed conditions under which the learning procedure converges to the true parameter values.[12] Hansen and Sargent (1998) use the Kalman filtering procedure to represent optimal learning in stochastic LR problems. They also show that the Kalman filtering procedure is the dual for the LR problem. As a result, the conditions for the convergence of the Kalman filter are similar to those for a stable LR problem. As a result, the perturbation is going to mimic the convergence behavior of the LR problem (optimal learning), since it approximates the optimal experimentation problem only in a neighborhood of this problem.

Under the perturbation method the optimal control solution cannot be separated from the experimentation problem since the first order perturbation of the optimal experimentation problem is dependent on the spread between the variance–covariance matrix under optimal experimentation and learning, respectively. In the analysis here the system of difference equations for both the closed loop system for the state vector and the equation of motion for the variance–covariance matrix are found in the neighborhood of the LR solution. Because the equation of motion for the variance–covariance matrix is not a function of the state vector, the system is upper triangular. This means that the conditions for the stability for the closed loop system and the Kalman filter are sufficient for the stability of the system.

In this paper the optimal experimentation problem is approximated by developing a parameterization which collapses the optimal experimentation problem to the optimal learning problem in LR problems. These conditions would rule out problems raised by Santos (1994). In addition, when beliefs converge, the expectation of the latent parameter will satisfy the mean prediction property.

[12]In optimal learning problems the agent's actions do not affect the information used to update forecast, while it does under optimal experimentation. See Evans and Honkapohja (2001) for a detail survey and discussion of the work on optimal learning. The recent special issue of this journal edited by Orphanides and Williams (2005) contains recent contributions to this literature.

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In addition, the LR problem will be optimized. The difference is that the Kalman gain will be smaller so that the conditional variance–covariance matrix will converge faster to its steady state value. Thus, the perturbation method applied to optimal experimentation problems in the neighborhood of the optimal learning problem will converge to correct limit beliefs as long as the LR problem is stable. Consequently, the approximation to the optimal experimentation problem will not capture the circumstances in which this problem displays non-differentiability of the value and policy functions. As a result, the analysis of Jin and Judd can be used to find the perturbation to the optimal experimentation problem only in the neighborhood of the LR problem.

This work complements the work on robust control by Hansen and Sargent (2001, 2003, 2004), Brock and Durlauf (2005), and Brock et al. (2003). In these papers different types of model misspecification are introduced into optimal control problems. In these papers the policy makers maximize their welfare subject to uncertainty about the correct model. This maximization is undertaken given that nature chooses the worst possible model. In the Hansen and Sargent framework model uncertainty is incorporated into the error specification of the true model. They show how the solution can be found using Riccati equations which are similar to those found in the LR problem. It should be feasible in future research to integrate the Hansen and Sargent model of robust control with the optimal experimentation in this work. Brock, Durlauf and West also consider more discrete model misspecification. For example, a RBC and a Neo Keynesian model of economy imply two distinct ways in which money growth will impact the economy. The model uncertainty in this paper is distinct from theirs and provides another reason why the policy maker would be cautious about the appropriate policy to undertake.

3. The LR problem

The perturbation method is used to solve a general experimentation problem by approximating the problem around a simpler problem with a known solution. In the optimal experimentation problem the LR problem is taken as the simpler problem. In this section the LR problem is summarized as well as its solution following Hansen and Sargent (1998), and Anderson et al. (1996). The agent is assumed to choose a sequence $\{u_t\}$ to maximize

$$-\mathrm{E}\left(\sum_{t=0}^{\infty}\beta^t[u_t'Ru_t + y_t'Q_{yy}y_t + 2y_t'Q_{yz}z_t + z_t'Q_{zz}z_t + 2u_t'W_y'y_t + 2u_t'W_z'z_t]|\mathscr{F}_0\right)$$

subject to

$$x_{t+1} \equiv \begin{pmatrix} y_{t+1} \\ z_{t+1} \end{pmatrix} = \begin{pmatrix} A_{yy} & A_{yz} \\ 0 & A_{zz} \end{pmatrix}\begin{pmatrix} y_t \\ z_t \end{pmatrix} + \begin{pmatrix} B_y \\ 0 \end{pmatrix}u_t + \begin{pmatrix} G_{yy} & G_{yz} \\ 0 & G_{zz} \end{pmatrix}\begin{pmatrix} w_{yt+1} \\ w_{zt+1} \end{pmatrix}$$
$$= Ax_t + Bu_t + Gw_{1t+1}.$$

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Here $\{\mathcal{F}_t : t = 0, \ldots\}$ is an increasing sequence of information sets which is based on a martingale difference process $w'_{1t+1} \equiv (w_{yt+1}, w_{zt+1})'$ such that $\mathrm{E}(w_{1t+1}|\mathcal{F}_t) = 0$ and $\mathrm{E}(w_{1t+1}w'_{1t+1}|\mathcal{F}_t) = I$. u_t is the $p \times 1$ control vector, which may influence the $q \times 1$ endogenous state vector y_t but does not effect the $r \times 1$ exogenous state vector, z_t. Each of the matrices are conformable to these vectors.

To solve this problem the cross product terms and the discount factor are eliminated by defining the selection matrices $U_y \equiv [I, 0]$ and $U_z \equiv [0, I]$ such that $U_z A U'_y = 0$, $U_z G U'_y = 0$, and $U_z B = 0$. Next let

$$y_t \equiv \beta^{t/2} U_y x_t, \quad z_t \equiv \beta^{t/2} U_z x_t, \quad v_t \equiv \beta^{t/2}(u_t + R^{-1}(W'_y \;\; W'_z)x_t), \tag{1}$$

$$\begin{pmatrix} A^0_{yy} & A^0_{yz} \\ 0 & A^0_{zz} \end{pmatrix} \equiv \beta^{1/2}(A - BR^{-1}W'), \quad B^0_y \equiv \beta^{1/2} U_y B, \text{ and}$$

$$\begin{pmatrix} Q^0_{yy} & Q^0_{yz} \\ Q^{0'}_{yz} & Q^0_{zz} \end{pmatrix} \equiv Q - WR^{-1}W'.^{13}$$

The solution to this LR problem is given by

$$v_t = -F_y y_t - F_z z_t,$$

where $F_y \equiv [R + B'_y P_y B_y]^{-1} B'_y P_y A_{yy}$ and $F_z \equiv [R + B'_y P_y B_y]^{-1} B'_y [P_y A_{yz} + P_z A_{zz}]$. The solution is found in two steps. First, P_y solves the Riccati equation

$$P_y = Q_{yy} + [A_{yy} - B_y F_y]' P_y [A_{yy} - B_y F_y] + F'_y R F_y$$

and second, P_z satisfies the Sylvester equation

$$P_z = Q_{yz} + [A_{yy} - B_y F_y]' P_y A_{yz} + [A_{yy} - B_y F_y]' P_z A_{zz}.$$

Reversing the definitions in (1) the solution to the discounted LR problem is

$$u_t = -[F_y + R^{-1}W'_y]y_t - [F_z + R^{-1}W'_z]z_t.$$

If we substitute this optimal decision rule into the law of motion for the state variables we obtain

$$\begin{aligned}\begin{pmatrix} y_{t+1} \\ z_{t+1} \end{pmatrix} &= \begin{pmatrix} A_{yy} - B_y[F_y + R^{-1}W'_y] & A_{yz} - B_y[F_z + R^{-1}W'_z] \\ 0 & A_{zz} \end{pmatrix} \\ &\quad \times \begin{pmatrix} y_t \\ z_t \end{pmatrix} + \begin{pmatrix} G_{yy} & G_{yz} \\ 0 & G_{zz} \end{pmatrix} \begin{pmatrix} w_{yt+1} \\ w_{zt+1} \end{pmatrix} \\ &= [A - BF]x_t + Gw_{1t+1}.\end{aligned}$$

This closed loop system determines the stability of the LR problem. In particular the eigenvalues of the matrix $[A - BF]$ must be all less than one for the system to be stable.[14]

[13] To avoid unnecessary notation I delete the superscript on A and Q below.

[14] See Anderson et al. (1996) and Ljungqvist and Sargent (2000) for a discussion of stability.

Under the LR problem the agent can learn about the economy independent of their optimal decisions. This result follows from the certainty equivalence property. Certainty equivalence is dependent on quadratic objectives, linear constraints and independence among the distribution of shocks and the agents choice.

In learning problems the agent observes signals which are a linear combination of the hidden state, control and random error vectors. For simplicity only the endogenous state vector is hidden from the agent. The endogenous state vector follows:

$$y_{t+1} = A_{yy}y_t + A_{yz}z_t + B_y u_t + G^1 w_{1t+1},$$ [15]

while the agent observes each period t the $s \times 1$ vector of signals

$$s_t = C_{sy}y_t + C_{sz}z_t + Du_t + Hw_{2t}.$$

Assume

$$\mathrm{E}\begin{pmatrix} w_{1t+1} \\ w_{2t} \end{pmatrix}(w'_{1t+1} \;\; w'_{2t}) = \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}.$$

The agent is interested in forecasting the endogenous state vector $\hat{y}_t = \mathrm{E}(y_t|u_t, z_t, s_t, s_{t-1}, \ldots, s_0, \hat{x}_0)$.[16] The Kalman filter updates the agent's forecast according to

$$\hat{y}_{t+1} = A_{yy}\hat{y}_t + A_{yz}z_t + B_y u_t + K_t a_t,$$

where $a_t = s_t - \hat{s}_t = C_{sy}(y_t - \hat{y}_t) + Hw_{2t}$. The Kalman Gain is defined by

$$K_t = A_{yy}\Sigma_t C'_{sy}(C_{sy}\Sigma_t C'_{sy} + HH')^{-1},$$

and the conditional variance–covariance matrix of the state is updated according to

$$\Sigma_{t+1} = A_{yy}\Sigma_t A'_{yy} + GG' - A_{yy}\Sigma_t C'_{sy}(C_{sy}\Sigma_t C'_{sy} + HH')^{-1}C_{sy}\Sigma_t A'_{yy}.$$

In the optimal experimentation literature the agent has some ability to manipulate the flow of information. This means that H is a function of the agent's decisions, so that the variance–covariance matrix for the signal is also a function of the control vector, u. As a result the agent's decision influences the distribution of the state vector. Thus, the certainty equivalence property no longer holds. This means that the agent's optimization problem cannot be separated from their forecasting problem. In the next section the optimal experimentation problem is formulated.

[15]The superscript 1 refers to the column's of G associated with the endogenous state variables. Without loss of generality from now on I will delete the superscript on G.

[16]Ljungqvist and Sargent (2000, pp. 643–649) derives the Kalman filter in a similar circumstances. They also demonstrate the duality between the optimal control and filtering problem.

4. Optimal experimentation

In the optimal experimentation problem the agent chooses a sequence $\{u_t\}$ to maximize

$$-\mathrm{E}\left(\sum_{t=0}^{\infty}\beta^t[u_t'Ru_t + y_t'Q_{yy}y_t + 2y_t'Q_{yz}z_t + z_t'Q_{zz}z_t + 2u_t'W_y'y_t + 2u_t'W_z'z_t]|\mathscr{F}_0\right)$$

subject to

$$x_{t+1} = Ax_t + Bu_t + Gw_{1t+1},$$

$$\hat{y}_{t+1} = A_{yy}\hat{y}_t + A_{yz}z_t + B_yu_t + K_ta_t = F[u_t, \hat{y}_t, z_t, \Sigma_t, \tau], \tag{2}$$

$$K_t = A_{yy}\Sigma_tC_{sy}'(C_{sy}\Sigma_tC_{sy}' + HH')^{-1}, \tag{3}$$

$$\begin{aligned}\Sigma_{t+1} &= A_{yy}\Sigma_tA_{yy}' + GG' - A_{yy}\Sigma_tC_{sy}'(C_{sy}\Sigma_tC_{sy}' + HH')^{-1}C_{sy}\Sigma_tA_{yy}' \\ &= G[u_t, z_t, \Sigma_t, \tau],\end{aligned} \tag{4}$$

$$z_{t+1} = A_{zz}z_t + G_{zz}w_{zt+1} = Z(z_t),$$

and

$$\mathrm{E}\left(\sum_{t=0}^{\infty}[|u_t|^2 + |y_t|^2]|\mathscr{F}_0\right) < \infty.$$

Under optimal experimentation the agent is learning about both the endogenous state vector y_t as well as some of the parameters of the model. Usually, these parameters are elements of the matrices A_{yy}, A_{yz} and B_y. Kendrick (2002a) keeps these two vectors separate in his notation since the control vector does not influence the parameters of the model. To keep the notation simpler the equation of motions for the unknown parameters is attached to the bottom of the endogenous state vector.

In the optimal experimentation problem, the variance–covariance of the signal, HH', is a function of the current control u_t and exogenous state vector z_t. In particular, the uncertainty in the signals is a linear function of the control and exogenous state vectors. This effect may be represented by replacing Hw_{2t} with

$$Hw_{2t} + \tau_1u_t'\varepsilon_{1t} + \tau_2z_t'\varepsilon_{2t},$$

where w_{2t}, ε_{1t}, and ε_{2t} are not correlated. The variance–covariance matrix, HH', is now

$$V_t = HH' + \tau_1^2u_t'V_3u_t + \tau_2^2z_t'V_4z_t, \tag{5}$$

where $V_3 = \mathrm{E}_t[\varepsilon_{1t}\varepsilon_{1t}']$ and $V_4 = \mathrm{E}_t[\varepsilon_{2t}\varepsilon_{2t}']$. The τ_1 and τ_2 are perturbation parameters such that each element is equal to one under optimal experimentation.[17] These

[17] Ljungqvist and Sargent (2000, pp. 643–649) allow for time varying variance–covariance matrix as long as the agent knows them.

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perturbation parameters can be thought of as the parameters in the definition of directional derivatives in the direction u_t and z_t, respectively. In addition, as both τ_1 and τ_2 approach zero the variance–covariance matrix approaches HH', so that the problem reduces to the LR problem.

In this case the Bellman equation becomes

$$V[\hat{y}_t, z_t, \Sigma_t, \tau] = \mathrm{E}[\Pi[u_t, y_t, z_t, \tau] + \beta V[\hat{y}_{t+1}, z_{t+1}, \Sigma_{t+1}, \tau] | \mathscr{F}_t]. \quad (6)$$

Here

$$\Pi[u_t, y_t, z_t, \tau] = -(u_t' R u_t + y_t' Q_{yy} y_t + 2y_t' Q_{yz} z_t + z_t' Q_{zz} z_t + 2u_t' W_y' y_t + 2u_t' W_z' z_t).$$

This dynamic programming problem incorporates two new effects which are not present in the LR problem. In both cases the control vector effects the variance–covariance of the signal by (5). The first effect measures the effect of the choice on the Kalman gain through Eq. (3) which in turn influences the conditional expectation of y_{t+1} in (2). The second effect deals with the optimal choice on the conditional variance–covariance matrix for y_{t+1} in (4). To analyze these effects it is necessary to find the first and second order derivatives of $F[u_t, \hat{y}_t, z_t, \Sigma_t, \tau]$ and $G[u_t, z_t, \Sigma_t, \tau]$ under optimal experimentation. In Appendix A the following Lemma is proved.

Lemma 1. *The partial derivative of the Kalman filter are* $\partial F/\partial u_t = vec([B_y', 0'])$, $\partial F/\partial \hat{y}_t = vec([A_{yy}', 0']) - (\binom{K_t}{0} \otimes I_q)(C_{sy} \otimes I_q)\, vec(I_q)$, $\partial F/\partial z_t = vec([A_{yz}', A_{zz}'])$, $\partial F/\partial \Sigma_t \neq 0$, $\partial F/\partial \tau_1 = 0$, $\partial F/\partial \tau_2 = 0$, $\partial^2 F/\partial u_t \partial \Sigma_t = 0$, $\partial^2 F/\partial \hat{y}_t \partial \Sigma_t \neq 0$, $\partial^2 F/\partial z_t \partial \Sigma_t = 0$, $\partial^2 F/\partial \Sigma_t^2 \neq 0$, $\partial G/\partial u_t = 0$, $\partial G/\partial z_t = 0$, $\partial G/\partial \Sigma_t \neq 0$, $\partial G/\partial \tau_1 = 0$, $\partial G/\partial \tau_2 = 0$, *and* $\partial^2 G/\partial \Sigma_t^2 \neq 0$ *when the perturbation parameters are zero.*[18]

The control and exogenous state variables change the agent's forecast, $F[u_t, \hat{y}_t, z_t, \Sigma_t, \tau]$, of the state vector only through their direct effect on the endogenous state vector. On the other hand, a change in the estimated endogenous state vector, $\hat{y}_t$, also changes the signal received by the agent which influences the state vector through the Kalman gain, K_t. The variance–covariance matrix effects the agent's forecast of the endogenous state vector through the change in the Kalman gain.

The equation of motion for the variance–covariance matrix, $G[u_t, z_t, \Sigma_t, \tau]$ is not influenced by the control and the exogenous state vector when the perturbation parameters are zero. The change in the current variance–covariance matrix on its future value has two effects. The direct effect is generally positive but the effect on the Kalman gain moves in the opposite direction. In the example discussed below the direct effect always dominates. Because the filtering problem is the dual of the control problem the variance–covariance matrix converges to a steady state over time under conditions similar to the conditions necessary for the stability of the state vector.

[18]To cut down on notation the functions F and Z have been stacked together and is called F. vec(A) stacks the columns of A in a vector starting with the first column.

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5. Perturbation method

The optimal experimentation problem introduced in the previous section does not have an explicit solution. In this section the perturbation method is used to approximate this problem following the analysis of Gaspar and Judd (1997), Judd (1998), and Jin and Judd (2002). The optimal experimentation problem fits into the general stochastic model of Jin and Judd. The perturbation of the stochastic model examined here is around the solution to the LR problem which is distinct from Jin and Judd in that they examine the perturbation around a certain steady state. However, the LR problem does have an analytic solution and the Kalman gain is analytic. As a result, the optimal experimentation problem should be analytic in the neighborhood of the LR problem, so that Jin and Judd's Theorem 6 implies that there is a unique and differentiable solution to the optimal experimentation problem in the neighborhood of the LR problem.[19]

The tensor notation is used extensively in the perturbation method.[20] This notation may be illustrated by writing the quadratic form $x'Ax$ as $a_{ij}x^iy^j$ which means $\sum_i \sum_j a_{ij}x^iy^j$. As a result, a summation occurs whenever a superscript and a subscript match. The partial derivatives $\partial F[\hat{y}_t, z_t, u_t, \Sigma_t, \tau]/\partial \hat{x}_t$ are represented by F^i_j for each state vector. For example $F^i_j = a^i_{zz,j}$ for the exogenous state vectors i and j. In a similar way F^i_α would be the partial derivative of the ith state variable with respect to the αth control. $F^i_\alpha = 0$ for the exogenous state vectors. F^i_I represents the partial derivative of the ith state variable with respect to the Ith variance or covariance term. $F^i_I = 0$ for the exogenous state vectors. Finally, $F^i_{\mathscr{I}}$ represents the partial derivative of the ith state variable with respect to the $\mathscr{I}$th perturbation parameter. $F^i_{\mathscr{I}} = 0$ for both state vectors.

Given this notation the Euler conditions may be written as

$$\begin{aligned} \mathrm{E}[\Pi_\alpha[u[\hat{y}_t, z_t, \Sigma_t, \tau], y_t, z_t, \tau] + \beta V_i[\hat{y}_{t+1}, z_{t+1}, \Sigma_{t+1}, \tau] F^i_\alpha[u[\hat{y}_t, z_t, \Sigma_t, \tau], \hat{y}_t, z_t, \Sigma_t, \tau] \\ + \beta V_I[\hat{y}_{t+1}, z_{t+1}, \Sigma_{t+1}, \tau] G^I_\alpha[u[\hat{y}_t, z_t, \Sigma_t, \tau], z_t, \Sigma_t, \tau] | \mathscr{F}_t] \leqslant 0 \end{aligned} \tag{7}$$

for each control $\alpha = 1 \cdots p$.

To illustrate how the tensor notation is used look at the second term for the αth control vector

$$\begin{aligned} &\beta V_i[\hat{y}_{t+1}, z_{t+1}, \Sigma_{t+1}, \tau] F^i_\alpha[u[\hat{y}_t, z_t, \Sigma_t, \tau], \hat{y}_t, z_t, \Sigma_t, \tau] \\ &\quad = \beta \sum_{i=1}^{q} \frac{\partial V[\hat{y}_{t+1}, z_{t+1}, \Sigma_{t+1}, \tau]}{\partial \hat{y}_{t+1i}} \frac{\partial \hat{y}_{t+1i}}{\partial u_\alpha}, \end{aligned}$$

where q is the dimension of the endogenous state vector. Consequently, this term is just the application of the chain rule for a higher dimension. It is also the usual term found in the necessary conditions for the optimal of a dynamic programming problem.

[19]See Chen et al. (2005) for a discussion of analytic solutions to linear integral equations.

[20]See Gaspar and Judd (1997) and Kay (1988) for details about using tensor notation.

The first two lines of (7) are similar to the Euler equation under the LR problem. The third line is new. It looks at changes in the future variance–covariance matrix as a result of a change in the control variable, $\partial\Sigma_{t+1,I}/\partial u_\alpha = G^I_\alpha$, which in turn influences the value of the firm, V_I. While the value function is affected by the variance–covariance matrix when the perturbation parameter is zero, Lemma 1 shows that the variance–covariance would not be changed by the change in the control variables in the neighborhood of the LR problem. Lemma 1 also shows that the variance–covariance matrix does not influence the effect of the control variable on the state variables, $\partial\hat{y}_{t+1i}/\partial u_\alpha = F^i_\alpha[u[\hat{y}_t, z_t, \Sigma_t, \tau], \hat{y}_t, z_t, \Sigma_t, \tau]$ in the neighborhood of the LR problem. This means that the main impact of the optimal experimentation under the perturbation is given by the impact of the future variance–covariance matrix on the marginal value of the state variable, $V_i[\hat{y}_{t+1}, z_{t+1}, \Sigma_{t+1}, \tau]$. As a result, $V_{iK}[\hat{y}^{\mathrm{LR}}_{t+1}, z_{t+1}, \Sigma^{\mathrm{LR}}_{t+1}, 0]$ must be calculated in the neighborhood of the LR problem where the superscript LR refers to the linear regulator solution.

Solving the optimal learning problem (6) and (7) explicitly is problematic. The difficulty comes about because of the additional non-linearity introduced by the control variables influence on the Kalman filter. However, the problem reduces to the LR problem when the perturbation parameter, τ, is set equal to zero. As a result, the perturbation method of Gaspar and Judd (1997), Judd (1998), and Jin and Judd (2002) may be applied to (6) and (7).

Theorem 6 of Jin and Judd applies to this situation with a slight modification. Jin and Judd apply the perturbation in the neighborhood of a certain steady state while we are interested in the neighborhood of the LR problem. However, we do know that the LR solution, $u[\hat{y}^{\mathrm{LR}}_t, z^{\mathrm{LR}}_t, \Sigma^{\mathrm{LR}}_t, 0]$, is analytical since it is a simple linear function of the state variables. It also follows from Lemma 1 that the equation of motion for the state vector, $F[u_t, \hat{y}_t, z_t, \Sigma_t, \tau]$, and the variance–covariance matrix, $G[u_t, z_t, \Sigma_t, \tau]$, can be differentiated to any order in the neighborhood of the LR solution, although Lemma 1 provides only the first two derivatives.[21] It also follows that these functions are analytical since they are compositions of analytic functions.[22] As a result, the argument of Jin and Judd follows so that there exist a unique solution, $u[\hat{y}_t, z_t, \Sigma_t, \tau]$ and $V[\hat{y}_t, z_t, \Sigma_t, \tau]$ to (6) and (7) in the neighborhood of the LR problem. In addition derivatives of this solution may be found through implicit differentiation in the same neighborhood.

The radius of convergence of this solution is not known. Up to now there are only limited results on the radius of convergence for the solution to integral equations such as (6) for the optimal experimentation problem. The radius of convergence is the maximum interval in which the Taylor series for the solution converges. Chen et al. (2005) develop a procedure for finding the radius of convergence to such an integral equation when the solution is in R^1 and the integral equation is linear in the

[21]Higher order perturbation of the control and value function are feasible, but it would be necessary to find the corresponding order of the derivatives of $F[u_t, \hat{y}_t, z_t, \Sigma_t, \tau]$ and $G[u_t, z_t, \Sigma_t, \tau]$. While this is feasible it would be very time consuming. As a result, one should rely on automatic differentiation to calculate higher order derivatives and perturbations. See www.autodiff.org.

[22]See Chapter 10 of Rudin (1974).

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solution. While more general results are not available, it is known that analyticity fails when the primitives are not analytical, the solution is not unique or the process is unstable.[23] In this case the primitives, such as the Kalman filter, are analytical. Following the work of Kiefer and Nyarko (1989), Balvers and Cosimano (1993), Keller and Rady (1999) and Wieland (2000a) the solution is unique as long as the experimentation effect is not too large. Finally, the conditions under which the process is stable will be discussed below. As a result, one can be confident the solution is within the radius of convergence for the problems discussed here, as long as one rules out these anomalies. In particular, the dynamic programming method of Wieland (2000a) can be used to identify the points of discontinuous policy functions and non-differentiable value functions in optimal experimentation problems.

The perturbation method involves a Taylor expansion of these functions around the known LR solution. In this case the expansion is around $[\hat{y}_t^{\mathrm{LR}}, z_t, \Sigma_t^{\mathrm{LR}}, 0]$.[24]

$$\begin{aligned} u^{\alpha}[\hat{y}_t, z_t, \Sigma_t, \tau] \approx{}& u^{\alpha}[\hat{y}_t^{\mathrm{LR}}, z_t, \Sigma_t^{\mathrm{LR}}, 0] + u_i^{\alpha}[\hat{y}_t^{\mathrm{LR}}, z_t, \Sigma_t^{\mathrm{LR}}, 0][\hat{x}_t - \hat{x}_t^{\mathrm{LR}}]^i \\ &+ u_I^{\alpha}[\hat{y}_t^{\mathrm{LR}}, z_t, \Sigma_t^{\mathrm{LR}}, 0][\Sigma_t - \Sigma_t^{\mathrm{LR}}]^I. \end{aligned} \tag{8}$$

$$\begin{aligned} V[\hat{y}_t, z_t, \Sigma_t, \tau] \approx{}& V[\hat{y}_t^{\mathrm{LR}}, z_t, \Sigma_t^{\mathrm{LR}}, 0] + V_i[\hat{y}_t^{\mathrm{LR}}, z_t, \Sigma_t^{\mathrm{LR}}, 0][\hat{x}_t - \hat{x}_t^{\mathrm{LR}}]^i \\ &+ V_I[\hat{y}_t^{\mathrm{LR}}, z_t, \Sigma_t^{\mathrm{LR}}, 0][\Sigma_t - \Sigma_t^{\mathrm{LR}}]^I \\ &+ \tfrac{1}{2} V_{ij}[\hat{y}_t^{\mathrm{LR}}, z_t, \Sigma_t^{\mathrm{LR}}, 0][\hat{x}_t - \hat{x}_t^{\mathrm{LR}}]^i[\hat{x}_t - \hat{x}_t^{\mathrm{LR}}]^j \\ &+ V_{iI}[\hat{y}_t^{\mathrm{LR}}, z_t, \Sigma_t^{\mathrm{LR}}, 0][\hat{x}_t - \hat{x}_t^{\mathrm{LR}}]^i[\Sigma_t - \Sigma_t^{\mathrm{LR}}]^I \\ &+ \tfrac{1}{2} V_{IJ}[\hat{y}_t^{\mathrm{LR}}, z_t, \Sigma_t^{\mathrm{LR}}, 0][\Sigma_t - \Sigma_t^{\mathrm{LR}}]^I[\Sigma_t - \Sigma_t^{\mathrm{LR}}]^J. \end{aligned} \tag{9}$$

As the perturbation vectors approach zero the variance–covariance term HH' approaches the LR value so that the equation of motion for the variance–covariance matrix, $G[u_t, z_t\Sigma_t, \tau]$, approaches its LR counterpart. Thus, Σ_t approaches Σ_t^{LR} as τ tends to zero. In addition, the Kalman Gain, K_t, also approaches its value under the LR problem so that $\hat{x}_t$ tends to $\hat{x}_t^{\mathrm{LR}}$. This means that the first two terms in (8) are identical to the LR problem so that

$$\begin{aligned} & u^{\alpha}[\hat{y}_t^{\mathrm{LR}}, z_t, \Sigma_t^{\mathrm{LR}}, 0] + u_i^{\alpha}[\hat{y}_t^{\mathrm{LR}}, z_t, \Sigma_t^{\mathrm{LR}}, 0][\hat{x}_t - \hat{x}_t^{\mathrm{LR}}]^i \\ &\quad = -[F_y + R^{-1}W_y]\hat{y}_t^{\mathrm{LR}} - [F_y + R^{-1}W_y][\hat{y}_t - \hat{y}_t^{\mathrm{LR}}] - [F_z + R^{-1}W_z]z_t. \end{aligned}$$

A similar argument applied to (9) leads to

$$\begin{aligned} & V[\hat{y}_t^{\mathrm{LR}}, z_t, \Sigma_t^{\mathrm{LR}}, 0] + V_i[\hat{y}_t^{\mathrm{LR}}, z_t, \Sigma_t^{\mathrm{LR}}, 0][\hat{x}_t - \hat{x}_t^{\mathrm{LR}}]^i \\ &\quad + \frac{1}{2} V_{ij}[\hat{y}_t^{\mathrm{LR}}, z_t, \Sigma_t^{\mathrm{LR}}, 0][\hat{x}_t - \hat{x}_t^{\mathrm{LR}}]^i[\hat{x}_t - \hat{x}_t^{\mathrm{LR}}]^j \\ &\quad = \rho + [\hat{x}_t^{\mathrm{LR}}]' \begin{pmatrix} P_y & P_z \\ P_z & P_{zz} \end{pmatrix} [\hat{x}_t^{\mathrm{LR}}] + [\hat{x}_t - \hat{x}_t^{\mathrm{LR}}]' \begin{pmatrix} P_y & P_z \\ P_z & P_{zz} \end{pmatrix} [\hat{x}_t - \hat{x}_t^{\mathrm{LR}}]. \end{aligned}$$

[23]Chen et al. (2005) provide an example of the first failure. Balvers and Cosimano (1993) provides an example of the second problem. Santos (1994) provides an example of the final case.

[24]In Appendix A it is shown that $V_{\mathscr{I}} = 0$ and $u_{\mathscr{I}}^{\gamma} = 0$.

Here P_{zz} was not needed for the solution to the LR problem but can be found from the Riccati equation following Hansen and Singleton (1998, p. 162.)

$$P_{zz} = Q_{zz} + A'_{yz}P_yA_{yz} + A'_{zz}P'_zA_{yz} + A'_{yz}P_zA_{zz} - [A'_{yz}P_yB_y + A'_{zz}P'_zB_y][R + B'_yP_yB_y]^{-1}[B'_yP_yA_{yz} + B'_yP_zA_{zz}] + A'_{zz}P_{zz}A_{zz},$$

which is solved by iterating on P_{zz}. Finally, the constant in the value function, ρ, is found by iterating on

$$\rho_{j+1} = \beta\rho_j + \beta\, trace(PGG').$$

This term represents the impact of uncertainty on the value of the agent under the LR problem so that uncertainty influences the value function even though it does not affect the decision rule. All these terms can be calculated efficiently using the doubling algorithm of Hansen and Sargent (1998).

The remainder of this section derives an expression for the last term in (8), $\partial u_\alpha/\partial\Sigma_{tI} = u^\alpha_I[\hat{y}^{\mathrm{LR}}_t, z_t, \Sigma^{\mathrm{LR}}_t, 0]$ where the second subscript on the variance–covariance matrix for the partial derivative refers to the particular element of the variance–covariance matrix. First the impact of the uncertainty on the value function is found by taking the total derivative of the value function (6) with respect to each of the q^2 variance–covariance terms in Σ_t. Here q is the number of endogenous state variables which are hidden from the agent.

$$V_I[\hat{y}_t, z_t, \Sigma_t, \tau] = \mathrm{E}_t[\beta V_j[\hat{y}_{t+1}, z_{t+1}, \Sigma_{t+1}, \tau]F^j_I[u[\hat{y}_t, z_t, \Sigma_t, \tau], \hat{y}_t, z_t, \Sigma_t, \tau] + \beta V_J[\hat{y}_{t+1}, z_{t+1}, \Sigma_{t+1}, \tau]G^J_I[u[\hat{y}_t, z_t, \Sigma_t, \tau], z_t, \Sigma_t, \tau]], \tag{10}$$

where $I = 1 \cdots q^2$ terms are ordered as in $vec(\Sigma_t)$. In this equation all the terms are known for the LR problem except V_I. Eq. (10) is a first order system of stochastic difference equations in the partial derivatives of the value function with respect to the variance–covariance matrix. The forcing term consist of the impact of the variance–covariance matrix on the state vector through the Kalman filter, F^j_I, which in turn changes the value function based on the marginal value of the state vector, V_j. The future impact of the variance–covariance matrix on the value function $V_J[\hat{y}_{t+1}, z_{t+1}, \Sigma_{t+1}, \tau]$ is changed because of the effect of the variance–covariance matrix on its future value, $\partial\Sigma_{t+1J}/\partial\Sigma_{tI} = G^J_I$. For stability the $q \times q$ matrix formed from G^J_I must have eigenvalues less than one. As a result, these eigenvalues being less than one is an essential condition for the perturbation to be well defined. In these circumstances, these equations can be stacked into a vector of q^2 first order linear difference equations which can be iterated on to yield $V_I[\hat{y}^{\mathrm{LR}}_t, z_t, \Sigma^{\mathrm{LR}}_t, 0]$ for all I.

To find the second order effects, $\partial V/\partial\Sigma_{tI}\partial y_{tk}$, in Appendix A (10) is differentiated with respect to the $q + r$ state variables. When the perturbation vector is set to zero, these derivatives are reduced to

$$V_{Ik}[\hat{y}^{\mathrm{LR}}_t, z_t, \Sigma^{\mathrm{LR}}_t, 0] = \mathrm{E}_t[\beta V_j[\hat{y}^{\mathrm{LR}}_{t+1}, z_{t+1}, \Sigma^{\mathrm{LR}}_{t+1}, 0]F^j_{Ik} + \beta V_{Jl}[\hat{y}^{\mathrm{LR}}_{t+1}, z_{t+1}, \Sigma^{\mathrm{LR}}_{t+1}, 0](F^l_k + F^l_\alpha u^\alpha_k)G^J_I]. \tag{11}$$

This equation is also a system of first order stochastic difference equations. In this case the forcing term is the effect of the variance–covariance matrix on the marginal impact of the current state vector on the future state vector, $\partial y_{t+1j}/\partial\Sigma_{tI}\partial y_{tk} = F^j_{Ik}$. The future impact of the variance–covariance matrix on the marginal value of the state vector, V_{Ik} is also changed because of the effect of the variance–covariance matrix on its future value, G^J_I. Again, the eigenvalues of the matrix made up of the elements G^J_I must be less than one for this system of difference equations to be stable. Consequently, if (11) are stacked together for each variance–covariance term, then (11) is a first order linear difference equation in the $q^2\times(q+r)$ terms for $V_{Ik}[\hat{y}^{\mathrm{LR}}_t, z_t, \Sigma^{\mathrm{LR}}_t, 0]$.

The results for a change in elements of the variance–covariance matrix on the optimal control, $\partial u_\alpha/\partial\Sigma_{tJ}$, can now be calculated. While the calculations are long, the results are simplified since the function G is not influenced by changes in the optimal controls when the perturbation vector is zero.[25] As a result, for each control vector $\gamma = 1\cdots p$, there are q^2 equations for each element of the variance–covariance matrix.

$$\begin{aligned} u^\gamma_J[\hat{y}^{\mathrm{LR}}_t, z_t, \Sigma^{\mathrm{LR}}_t, 0] = & -[\mathrm{E}_t[\Pi_{\alpha,\gamma}[u^{\mathrm{LR}}_t, y^{\mathrm{LR}}_t, z_t, 0] + \beta V_{ik}[\hat{y}^{\mathrm{LR}}_{t+1}, z_{t+1}, \Sigma^{\mathrm{LR}}_{t+1}, 0]F^i_\alpha F^k_\gamma]]^{-1} \\ & \times[\mathrm{E}_t[\beta V_{iK}[\hat{y}^{\mathrm{LR}}_{t+1}, z_{t+1}, \Sigma^{\mathrm{LR}}_{t+1}, 0]F^i_\alpha G^K_J]], \end{aligned} \tag{12}$$

where the inverse refers to the inverse tensor matrix.[26] This system of equations may be formulated as a system of $p\times q^2$ equations which can be solved for all the partial derivatives $\partial u_\alpha/\partial\Sigma_{tJ}$. When the matrix of this system of equations is negative definite, then the problem has an interior solution so that the bang–bang solution of Balvers and Cosimano (1993), etc. is not present in the approximation. This condition is identical to the condition in Theorem 6 of Jin and Judd (2002) that the operator is invertible in the neighborhood of the LR problem.

By substituting (12) into (8) the linear approximation of the optimal controls is complete. Examination of (12) reveals that the variance–covariance matrix for the hidden state variables influences the decisions of the agent through its influence on the value function for the agent's problem. The chain rule implies that there are two parts to this effect. First, the change in uncertainty changes the variance–covariance matrix through the Kalman filter, G^K_J. Next the variance–covariance matrix impacts the evaluation of how the control vector influences the marginal future value of the state vector, $V_{iK}F^i_\alpha$. Both of these effects are manifested through the change in uncertainty on the marginal value of the state vector, V_{iK} based on (11). These effects work through the impact on the value function in this respect the results are similar to Ito's lemma in continuous time stochastic control.

[25]Eq. (12) is derived in Appendix A.

[26]See Judd (1998, p. 500).

The stability of the optimal experimentation problem can also be examined by substituting the optimal solution (8) into the equation of motion for the state vectors.

$$\begin{pmatrix} y_{t+1} \\ z_{t+1} \\ vec[\Sigma_{t+1} - \Sigma^{\mathrm{LR}}_{t+1}] \end{pmatrix}$$
$$= \begin{pmatrix} A_{yy} - B_y[F_y + R^{-1}W'_y] & A_{yz} - B_y[F_z + R^{-1}W'_z] & B_y u^{\gamma}_J \\ 0 & A_{zz} & 0 \\ 0 & 0 & G^J_I \end{pmatrix}$$
$$\times \begin{pmatrix} y_t \\ z_t \\ vec[\Sigma_t - \Sigma^{\mathrm{LR}}_t] \end{pmatrix} + \begin{pmatrix} G_{yy} & G_{yz} \\ 0 & G_{zz} \\ 0 & 0 \end{pmatrix} \begin{pmatrix} w_{yt+1} \\ w_{zt+1} \end{pmatrix}.$$

Here both elements u^{γ}_J and G^J_I are evaluated at $[\hat{y}^{\mathrm{LR}}_{t+1}, z_{t+1}, \Sigma^{\mathrm{LR}}_{t+1}, 0]$ and are stacked in conformable matrices. The last line is added to the original closed loop system of the LR problem, $x_{t+1} = [A - BF]x_t + Gw_{1t+1}$, because the variance–covariance matrix now influences the optimal decision and is part of the state vector. This last line is the first order approximation of the equation of motion for the variance–covariance matrix, G, in the neighborhood of the LR problem. From the stability analysis of the LR problem the conditions for $[A - BF]$ being stable are already known. Now the new matrix for the closed loop system is a triangular matrix. As a result, the only additional conditions are that the eigenvalues of the matrix formed by G^J_I need to be less than one which was used to assure the stability of (10) and (11). However, Ljungqvist and Sargent (2000, pp. 650–652) demonstrate that the Kalman filter is the dual to the LR problem. Thus, a similar set of conditions assure that the closed loop for the optimal experimentation problem in the neighborhood of the LR problem is stable.

In summary, the perturbation of the optimal experimentation problem in the neighborhood of the LR problem is well defined. First, we know the primitives such as the Kalman Filter are analytical. Second, we know the solution is unique as long as the matrix formed by the elements $[\mathrm{E}_t[\Pi_{\alpha,\gamma}[u^{\mathrm{LR}}_t, y^{\mathrm{LR}}_t, z_t, 0] + \beta V_{ik}[\hat{y}^{\mathrm{LR}}_{t+1}, z_{t+1}, \Sigma^{\mathrm{LR}}_{t+1}, 0]F^i_{\alpha}F^k_{\gamma}]]$ is negative definite in the neighborhood of the LR problem. Finally, we know that the closed loop system of the optimal experimentation problem in the neighborhood of the LR problem is stable as long as the matrix $[A - BF]$ is stable. Thus, (8) represents the first order approximation of the optimal experimentation problem in the neighborhood of the LR problem.

6. An example

Balvers and Cosimano (1990) use optimal experimentation to explain the slow adjustment of prices for a firm with some monopoly power. Subsequently, Cosimano et al. (2002) apply this argument to a bank to explain the slow adjustment of loan and deposit rates to changes in the treasury bill rate. The presence of monopoly power can be rationalized based on switching cost for the bank's customers along the lines of Klemperer (1995). The switching cost implies that the demand for loans is dependent on market share. The market share is represented by the presence of lagged loans and deposits in the demand for loans and supply of deposits, respectively. The demand for loans is also dependent on the bank's rate relative to the average loan rate in the market. The supply of deposits is also dependent on the relative deposit rate.[27]

These generalizations of the original Balvers and Cosimano model yields an optimal experimentation problem in which there are two control variables, eleven state variables including lagged dependent variables, and two signals used to estimate four parameters. Yet, the solution can be approximated quickly on a standard computer.

The bank sees the demand for loans

$$L_t = l_{0,t} + l_1 L_{t-1} - l_{2,t}[r^{\mathrm{L}}_{t,i} - r^{\mathrm{L}}_t] + \varepsilon_{t,1}$$

and the supply of deposits

$$D_t = d_{0,t} + d_1 D_{t-1} + d_{2,t}[r^{\mathrm{D}}_{t,i} - r^{\mathrm{D}}_t] + \varepsilon_{t,2}.$$

Here, define L_t as the demand for loans by the ith bank at time t; $\varepsilon_{t,1}$ is the random change in the demand for loans for the ith bank at time t[28]; $r^{\mathrm{L}}_{t,i}$ is the ith bank's loan rate at time t; $r^{\mathrm{L}}_t \equiv (1/(N-1))\sum_{j=1,j\neq i}^{N} r^{\mathrm{L}}_{t,j}$ is the average loan rate in the bank's market at time t excluding this institution, where N is the number of competitor banks; D_t represents the supply of deposits to the ith bank at time t; $r^{\mathrm{D}}_{t,i}$ is the ith bank's deposit rate at time t; $\varepsilon_{t,2}$ is the random change in the supply of deposit for the ith bank at time t; $r^{\mathrm{D}}_t \equiv (1/(N-1))\sum_{j=1,j\neq i}^{N} r^{\mathrm{D}}_{t,j}$ is the average deposit rate in the bank's market at time t excluding this institution.

The bank observes the quantity of loans and deposits but does not know the true slope and intercepts for the demand for loans and supply of deposits. The intercepts are autoregressive to represent the consumers who are not sensitive to changes in interest rates. In particular, $l_{0,t} = l_0 + a_{11} l_{0,t-1} + \varepsilon_{t,3}$ and $d_{0,t} = d_0 + a_{22} d_{0,t-1} + \varepsilon_{t,4}$, while the slopes are given by $l_{2,t} = l_2 + \varepsilon_{t,5}$ and $d_{2,t} = d_2 + \varepsilon_{t,6}$. As a result, the bank sees the two signals

$$s_1 = l_{0,t} - \tau_L \varepsilon_{t,5}(r^{\mathrm{L}}_{t,i} - r^{\mathrm{L}}_t) + \varepsilon_{t,1} \quad \text{and} \quad s_2 = d_{0,t} - \tau_D \varepsilon_{t,6}(r^{\mathrm{D}}_{t,i} - r^{\mathrm{D}}_t) + \varepsilon_{t,2}.$$

[27]See Varian (1980) for a model of monopolistically competitive market in which a distribution in price represents an equilibrium strategy. The model only represents the decision problem of an individual bank for simplicity. McGrattan (1994) could be used to introduce strategic considerations but this would be beyond the scope of this paper.

[28]The normality assumption cannot be strictly true since the demand for loans could be negative. To avoid this possibility the normality assumption could be dropped, as long as, the bank cares about the mean and variance of the state variables. See Ljungqvist and Sargent (2000) for the derivation of the Kalman Filter under this case.

The bank chooses loan and deposit rates which maximize profits per period:

$$(r^{\mathrm{L}}_{t,i} - r_t - C^{\mathrm{L}}_t)L_t + (r_t(1-\alpha) - r^{\mathrm{D}}_{t,i} - C^{\mathrm{D}}_t)D_t,$$

subject to the demand for loans and the supply of deposits. The r_t is the treasury bill rate; α is the reserve ratio and C^{L}_t, C^{D}_t are the marginal resource cost of loans and deposits, respectively. These marginal resource costs are assumed constant for simplicity. The control vector is $u_t \equiv (r^{\mathrm{L}}_{t,i}\ r^{\mathrm{D}}_{t,i})'$, the endogenous state vector is $y_t \equiv (C^{\mathrm{L}}_t\ C^{\mathrm{D}}_t\ L_{t-1}\ D_{t-1}\ l_{0,t}\ d_{0,t})'$ and the exogenous state vector is $z_t \equiv (r^{\mathrm{L}}_t\ r^{\mathrm{D}}_t\ r^{\mathrm{C}}_t\ r_t\ 1)'$. The equations of motion for the parameters, $l_{0,t}$ and $d_{0,t}$, the bank wants to learn are included in the endogenous state vector. The interest rate on checking accounts, r^{C}_t is added to the vector of exogenous variables since it is statistically significant in the VAR estimates. The matrices in the augmented LR problem are

$$R \equiv \begin{pmatrix} l_2 & 0 \\ 0 & d_2 \end{pmatrix}; \quad W'_z \equiv \begin{pmatrix} -l_2 & 0 & 0 & -l_2 & 0 \\ 0 & -d_2 & 0 & -d_2(1-\alpha) & 0 \end{pmatrix};$$

$$W'_y \equiv \begin{pmatrix} -l_2 & 0 & -l_1 & 0 & -1 & 0 \\ 0 & d_2 & 0 & d_1 & 0 & 1 \end{pmatrix};$$

A_{zz} has roots less than one;

$$Q_{yy} \equiv \frac{1}{2}\begin{pmatrix} 0 & 0 & l_1 & 0 & 1 & 0 \\ 0 & 0 & 0 & d_1 & 0 & 1 \\ l_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & d_1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \end{pmatrix}; \quad Q_{zz} \equiv \frac{1}{2}\begin{pmatrix} 0 & 0 & 0 & l_2 & 0 \\ 0 & 0 & 0 & 0 & d_2(1-\alpha) \\ 0 & 0 & 0 & 0 & 0 \\ l_2 & 0 & 0 & 0 & 0 \\ 0 & d_2(1-\alpha) & 0 & 0 & 0 \end{pmatrix};$$

$$Q_{yz} \equiv \frac{1}{2}\begin{pmatrix} l_2 & 0 & 0 & 0 & 0 \\ 0 & -d_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & l_1 & 0 \\ 0 & 0 & 0 & -d_1(1-\alpha) & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -(1-\alpha) & 0 \end{pmatrix}; \quad A_{yy} \equiv \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & l_1 & 0 & 1 & 0 \\ 0 & 0 & 0 & d_1 & 0 & 1 \\ 0 & 0 & 0 & 0 & a_{11} & 0 \\ 0 & 0 & 0 & 0 & 0 & a_{22} \end{pmatrix};$$

$$A_{yz} \equiv \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ l_2 & 0 & 0 & 0 & 0 \\ 0 & -d_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & l_0 \\ 0 & 0 & 0 & 0 & d_0 \end{pmatrix}; \quad B_y \equiv \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ -l_2 & 0 \\ 0 & d_2 \\ 0 & 0 \\ 0 & 0 \end{pmatrix};$$

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$$G_{yy} \equiv \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \sigma_1 & 0 & \sigma_3 & 0 \\ 0 & 0 & 0 & \sigma_2 & 0 & \sigma_4 \\ 0 & 0 & 0 & 0 & \sigma_3 & 0 \\ 0 & 0 & 0 & 0 & 0 & \sigma_4 \end{pmatrix};$$

$$C_{sy} \equiv \begin{pmatrix} 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}; \quad C_{sz} \text{ and } D \text{ are zero.}$$

$$H \equiv \begin{pmatrix} \sigma_1 & 0 \\ 0 & \sigma_2 \end{pmatrix}, \quad V_3 = V_4 \equiv \begin{pmatrix} \sigma_5^2 & 0 \\ 0 & \sigma_6^2 \end{pmatrix},$$

$$V_5' \equiv \begin{pmatrix} \sigma_1^2 & 0 & 0 & 0 & 0 & 0 \\ 0 & \sigma_2^2 & 0 & 0 & 0 & 0 \end{pmatrix}.^{29}$$

The parameter values for this model are listed in Table 1. The parameter estimates are based on monthly data from 1993 to 1999 which was taken from a financial institution in a large metropolitan area.[30] Loan commitments are used for the loan demand and savings accounts are used for deposits. Both accounts are highly persistent with the expected dependence on the spread between bank rates and market rates for the metropolitan area. The exogenous state variable, z_t, is represented by the VAR model in Tables 2 and 3 which are used to construct the matrices A_{zz} and G_{zz}. This state vector includes the market rates for loan commitments-r^{L}, savings deposit-r^{D}, interest bearing checking accounts-r^{C}, treasury bill rate-r and the constant. One month lag in each of the interest rate variables was sufficient to generate white noise errors.

The first step of the simulation procedure is to implement the doubling algorithm of Hansen and Sargent (1998) and Anderson et al. (1996). This procedure generates the LR solution. The behavior of the bank's loan rate relative to the market loan rate is portrayed in Fig. 1 by the squares while the triangles represent the market rates. The bank's deposit rate over 30 months is given in Fig. 2. For these simulations 11 random draws from the normal distribution each month were used to generate the movement in the state vector, x_t from month to month. The first month is an initial value for the state vector which approximates a steady state. In Table 4 the state vectors for the first and sixth month are recorded for the LR problem in columns 2 and 4. The optimal loan and deposit rate for the bank are listed in Table 5 in

[29]The stochastic specification is slightly different since there is a variance–covariance matrix, V_5, between w_{1t+1} and w_{2t}. This causes the Kalman filter to change by replacing $A\Sigma_t C'$ with $A\Sigma_t C' + V_5$.

[30]Rich Sheehan provided these estimates.

Table 1
Parameters

Variable	Value	Standard deviation	Value
l_0	$4.8601 * 10^4$	σ_1	$1.4436 * 10^6$
a_{11}	0.9000	σ_2	$1.2017 * 10^7$
d_0	$2.1029 * 10^5$	σ_3	$5.9955 * 10^5$
a_{22}	0.9000	σ_4	$7.4062 * 10^5$
l_1	0.9946	σ_5	$1.8423 * 10^8$
d_1	0.9245	σ_6	$2.1667 * 10^9$
l_2	$9.0943 * 10^8$		
d_2	$4.3335 * 10^9$		
α	0		

Table 2
VAR for exogenous state vector

State variable	r^L	r^D	r^C	r	Constant
r^L	0.5512	−0.5415	0.5000	0.2785	0.0470
r^D	0.0172	0.6793	0.4354	0.0367	−0.00218
r^C	−0.01459	0.0527	0.9075	0.0400	−0.0006
r	0.0352	−0.9238	1.2885	0.8467	0.0094

Table 3
Variance–covariance matrix for VAR

State variable	r^L	r^D	r^C	r
r^L	$2.0177 * 10^{-5}$	$5.5679 * 10^{-7}$	$1.1708 * 10^{-7}$	$4.4902 * 10^{-9}$
r^D		$1.2331 * 10^{-7}$	$-1.1915 * 10^{-8}$	$-5.5679 * 10^{-10}$
r^C			$8.2302 * 10^{-7}$	$1.0038 * 10^{-9}$
r				$3.27 * 10^{-6}$

columns 2 and 4. The loan rates under both learning and experimentation start above the market loan rate. The loan rate is about 51 basis points below the market average rate in the sixth month, while the deposit rate is about 1.6% above the market average after six months. As a result, there is a tendency for the bank to set its loan rate below the market rate and the deposit rate above the market rate so as to expand the bank's value even though there is no experimentation.

In Table 6 column's 2 and 4 and Fig. 3 the Kalman gain under learning starts at about 0.5 for the loan rate and after six months it decreases to near the steady state, 0.3. This decline in the Kalman gain follows from the decline in the conditional

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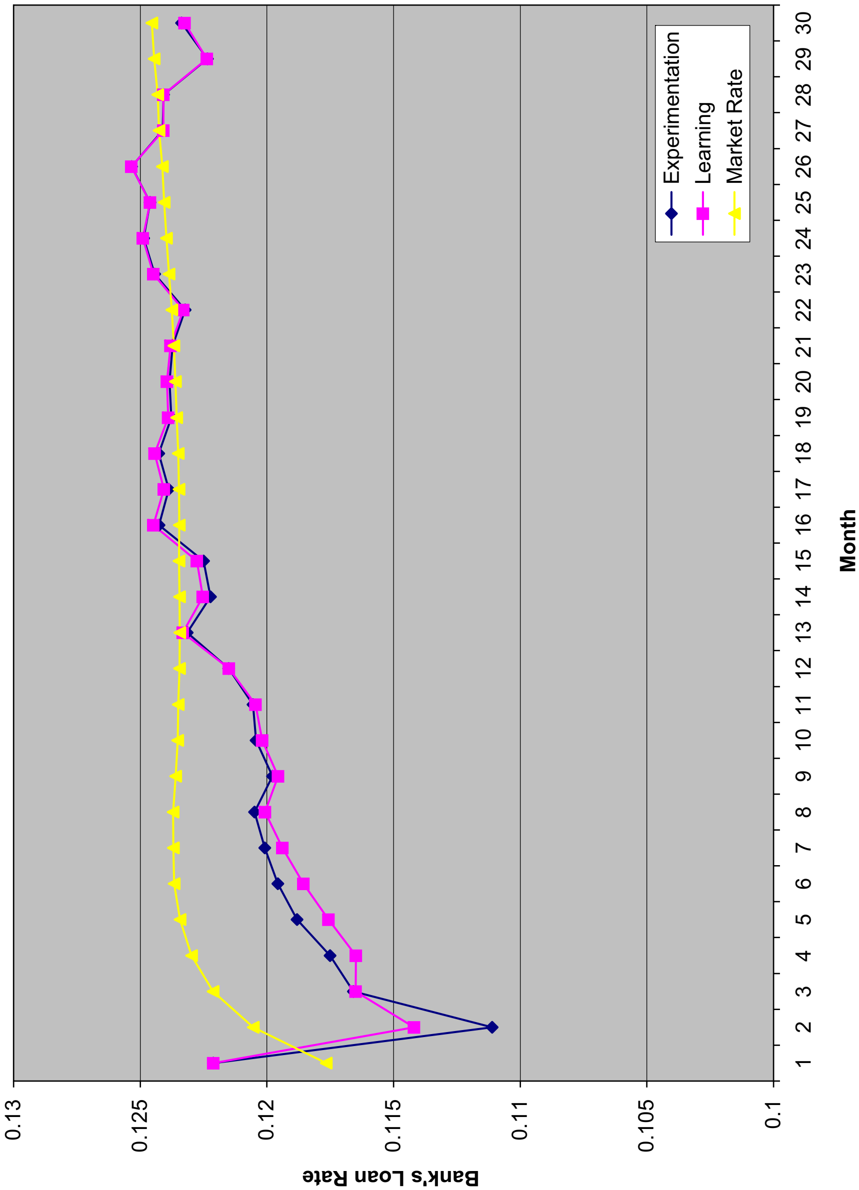

Fig. 1.

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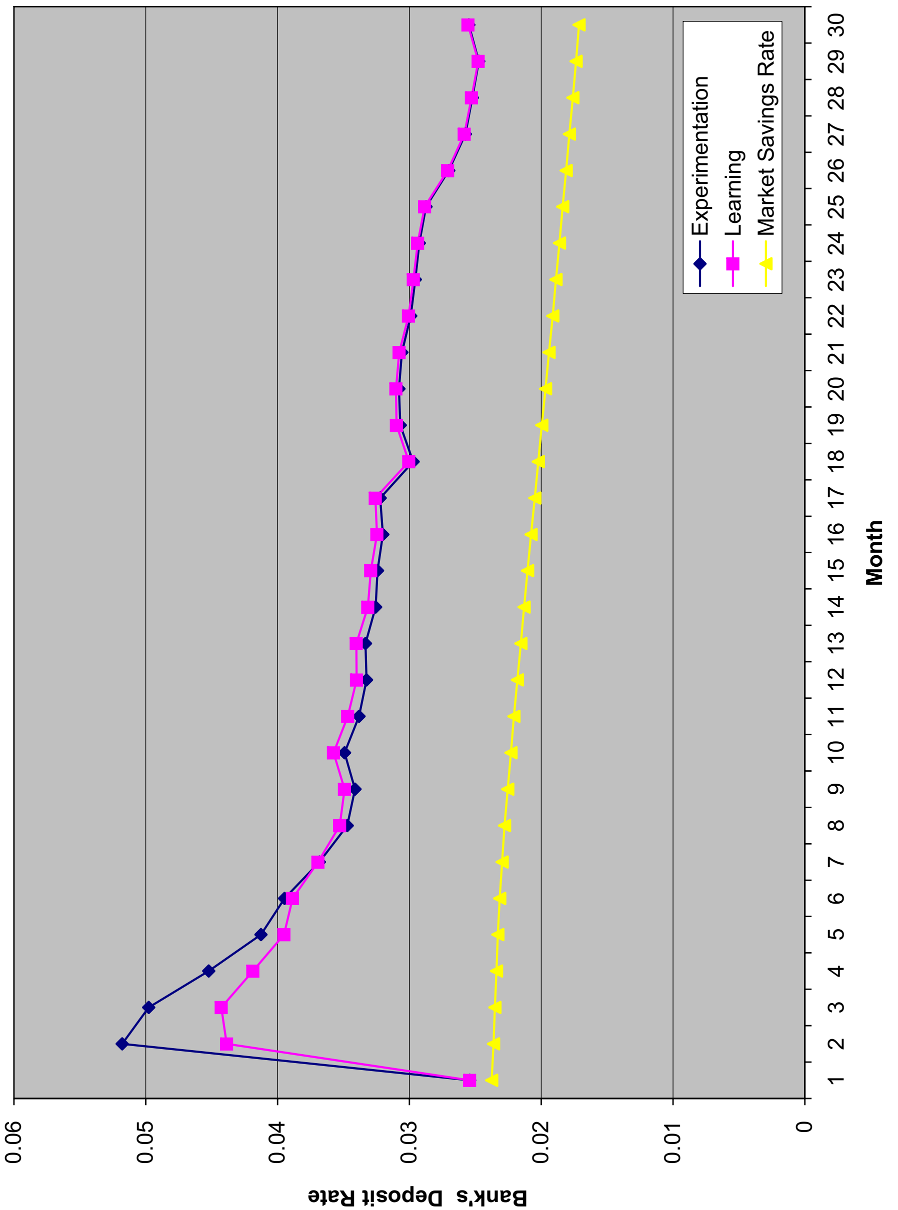
Bank's Deposit Rate
0.06
0.05
0.04
0.03
0.02
0.01
0
Month
1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 21 22 23 24 25 26 27 28 29 30
Experimentation
Learning
Market Savings Rate

Fig. 2.

Table 4
State vector for 1st and 6th month

State variable	x_1	x_1^{LR}	x_6	x_6^{LR}
C^L	0.0600	0.0600	0.0600	0.0600
C^D	0.0100	0.0100	0.0100	0.0100
L	$4.4424 * 10^7$	$4.4424 * 10^7$	$7.2433 * 10^7$	$7.1816 * 10^7$
D	$4.4062 * 10^7$	$4.4062 * 10^7$	$2.6379 * 10^8$	$1.9506 * 10^8$
$l_{0,t}$	$1.3108 * 10^6$	$1.3108 * 10^6$	$5.8149 * 10^6$	$5.9149 * 10^6$
$d_{0,t}$	$-5.9325 * 10^7$	$-5.9325 * 10^7$	$-3.1834 * 10^7$	$-3.3592 * 10^7$
r^L	0.1177	0.1177	0.1237	0.1237
r^D	0.0238	0.0238	0.0231	0.0231
r^C	0.0129	0.0129	0.0122	0.0122
r	0.0542	0.0542	0.0538	0.0538
Constant	1	1	1	1

Table 5
Control vector for 1st and 6th month

Control variable	u_1	u_1^{LR}	u_6	u_6^{LR}
r_i^L	0.1221	0.1221	0.1196	0.1186
r_i^D	0.0254	0.0254	0.0394	0.0389

Table 6
Kalman gain for 1st and 6th month

Kalman gain	K_1	K_1^{LR}	K_6	K_6^{LR}
r_i^L	0.1828	0.4956	0.2591	0.2911
r_i^D	0.0063	0.4487	0.0016	0.0774

variance of the loan intercept which can be seen in Fig. 4.[31] As the conditional variance of the constant for loans decreases over the six months, as seen in Table 7 row 2 or Fig. 4, the Kalman gain decreases which increases the convergence of this conditional variance. The deposit rate exhibits similar patterns for the Kalman filter accept that the change is more pronounce.

[31]The initial value of the Kalman gain can be manipulated by changing the variance of the constant relative to the variance in the regression. By lowering the variance of the constant the conditional variance of the constant decreases relative to the variance of the regression which leads to a decrease in the Kalman gain.

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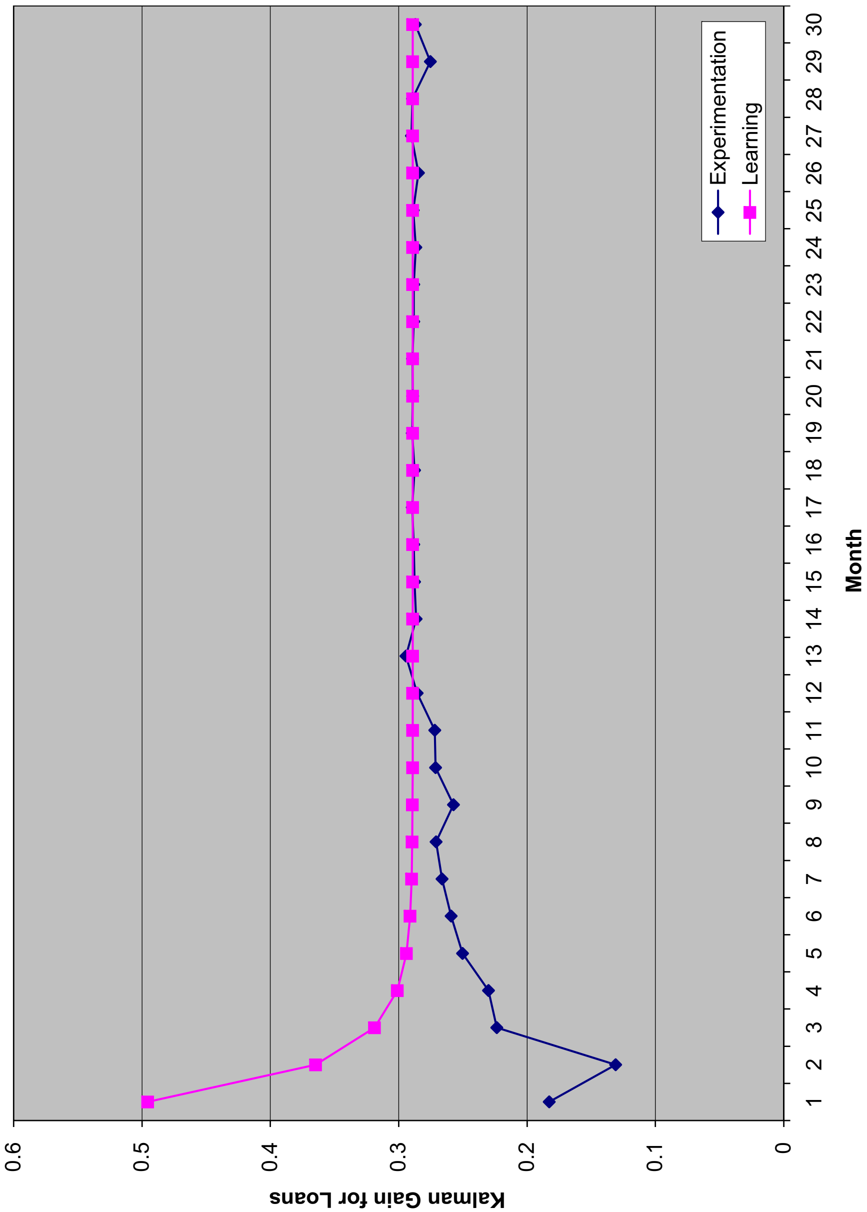

Fig. 3.

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Table 7
Variance for 1st and 6th month

	Σ_1^{L}	Σ_1^{D}	Σ_6^{L}	Σ_6^{D}
Optimal	$6.1705 * 10^{11}$	$9.9127 * 10^{12}$	$9.2829 * 10^{11}$	$2.2201 * 10^{12}$
LR	$2.0474 * 10^{12}$	$1.1751 * 10^{13}$	$8.5563 * 10^{11}$	$1.2122 * 10^{13}$

In Table 8 the marginal value of the state variable, V_i is listed. These results are consistent with intuition. An increase in cost reduces the value of the bank, while an increase in demand increases its value. In summary, the LR solution behaves in a consistent and intuitive way.

The conditional variance has a positive impact on the value of the bank in Table 9. This results from the difference equation (10). The source of the positive sign is the impact of the variance–covariance matrix on the Kalman Gain, F_I^j. Balvers and Cosimano (1990) show that the bank has an increasing return to uncertainty in the intercept. As a result, Jensen's inequality implies that higher uncertainty leads to an increase in the value of the bank, $V_I > 0$.

We also can find the effect of the variance–covariance matrix on the marginal value of the state vector to the bank, V_{kI}. From the difference equation (11), the sign of this second derivative is driven by F_{kI}^j which represents the impact of the variance–covariance matrix on the marginal effect of the state on the Kalman Gain. Suppose we look at the diagonal elements of the variance–covariance matrix to identify the sign of this second order partial derivative. A higher expected value of the current intercept reduces the value of the signal observed today and this signal is now multiplied by a larger Kalman gain because of the higher conditional variance of the intercept. As a result, the diagonal elements of the variance–covariance matrix have a negative impact on the marginal value of the state vector, $V_{kI} < 0$. Of course, the diagonal elements could be any sign depending on the covariance between the signals.

We can now examine the impact of optimal experimentation on the behavior of the bank. The simulation starts at the same initial values so that the first occurrence of experimenting is in month 2. This behavior is pictured in Figs. 1 and 2 for the loan and deposit rates, respectively. The rhombus represents the optimal experimentation. The loan rate initially goes below the benchmark LR solution by 31 basis points and goes about nine basis points above by the sixth month. As a result, the bank initially increases market share with a lower loan rate which the bank takes advantage of later in time. This decrease in the loan rate increases the uncertainty in the regression since the loan rate is further below the market rate. Thus, the Kalman Gain decreases to 0.13 in the second month and converges to the LR value by the 12th month as seen in Fig. 3. Consequently, the conditional variance for the loan intercept converges faster to the steady state. The same experimentation occurs for the deposit rate except that the deposit rate is higher since it is a source of cost rather than revenue. In addition, the experimentation effect is larger for the deposit rate since the elasticity of supply for deposits is larger.

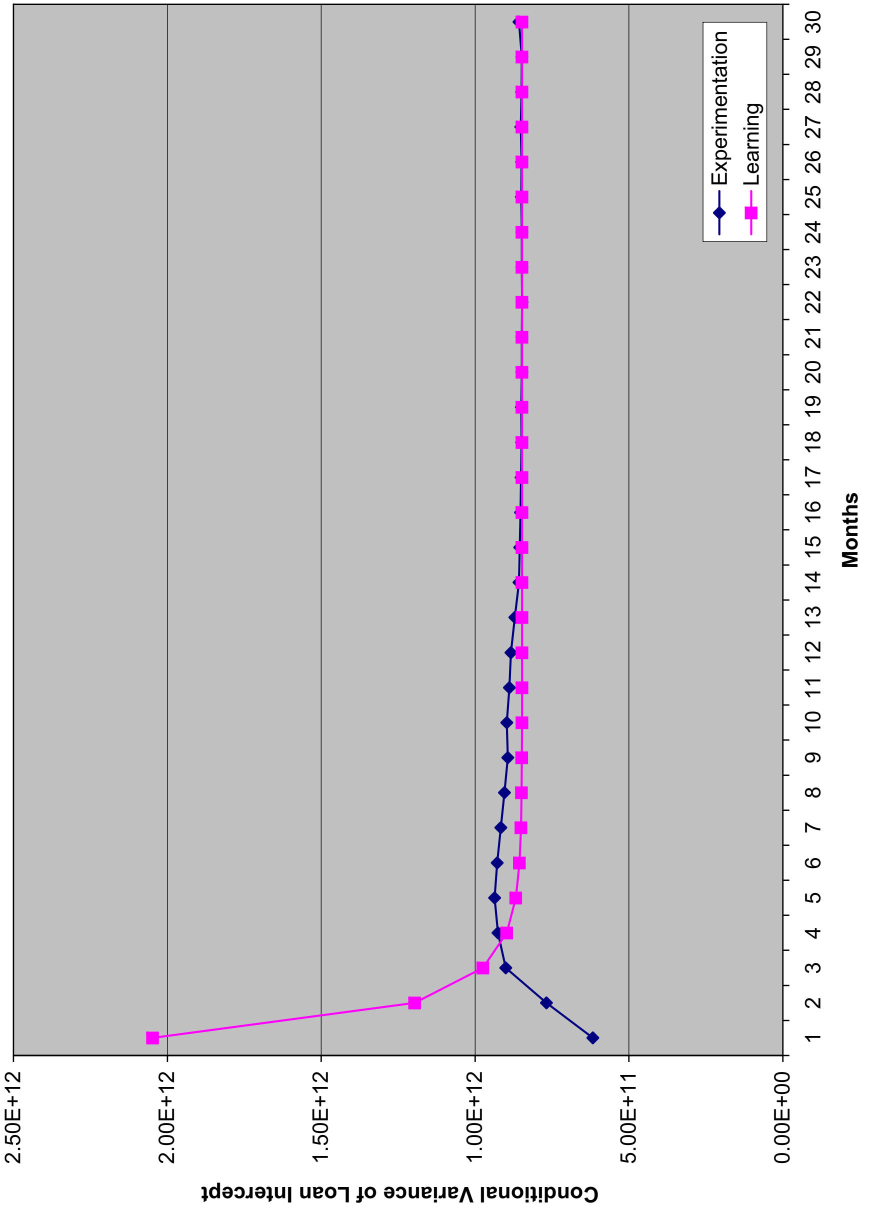

Fig. 4.

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Table 8
Partial derivative of value function with respect to state vector

State variable	V_i
C^{L}	$-1.2817 * 10^9$
C^{D}	$-5.5622 * 10^9$
L	0.1294
D	0.1114
$l_{0,t}$	0.7153
$d_{0,t}$	0.6548
r^{L}	$7.8314 * 10^7$
r^{D}	$-4.2614 * 10^9$
r^{C}	$-1.0526 * 10^9$
r	$1.3554 * 10^9$
Constant	$2.4840 * 10^8$

Table 9
V_I for 1st and 6th month

	1st Month	6th Month
Σ_1^{L}	$1.2730 * 10^{-9}$	$7.5818 * 10^{-10}$
Σ_1^{D}	$4.9437 * 10^{-10}$	$4.9505 * 10^{-10}$

In Table 10 we can see the effect of the conditional uncertainty on the optimal decisions of the bank in (12). The increase in the conditional variance of the intercept leads to a higher conditional variance in the future, $G_J^K > 0$. The higher conditional variance leads to a decrease in the marginal impact of the state vector on the value function, $V_{iK} < 0$ by (11). This change in the marginal value of the state vector together with the reduction in the demand for loans because of the higher loan rate, $F_\alpha^i < 0$ leads to an increase in the loan rate charged by the bank. Thus, the bank lowers the loan rate, see Fig. 1, when the conditional variance of the intercept under experimentation is below its value under the LR value, see Fig. 4.

The path of the loan rate in Fig. 1 can now be understood. The conditional variance of the demand for loan intercept under experimentation is initially below the LR case in Fig. 4 so that the loan rate is lower. By the sixth month this conditional variance is higher under optimal experimentation and its marginal impact on the optimal loan rate has also fallen. As a result, the loan rate is now slightly above the LR case. This continues till the 20th month when the loan rate is roughly the same as in the LR case.

The same basic pattern occurs for the deposit rate. The deposit rate first goes above the LR case by 79 basis points since higher deposit rates increases the supply of deposits. By the sixth month the spread is down to six basis points. Again by the 20th month there is no difference between the learning and experimentation decision.

Table 10
u_J^{γ} for 1st and 6th month

	Σ_1^{L}	Σ_1^{D1}	Σ_6^{L}	Σ_6^{D1}
r_i^{L}	$3.4791 * 10^{-15}$	0	$1.1589 * 10^{-14}$	0
r_i^{D}	0	$-5.3098 * 10^{-17}$	0	$-5.4152 * 10^{-16}$

These simulations confirm most of the qualitative results of Balvers and Cosimano (1990). The main new result is that after experimenting for a short period of time, the optimal loan rate goes above the loan rate under optimal experimenting. This new insight occurs since it is now possible to analyze optimal experimentation problems in a setting with changing state variables as well as more complex dynamic LR problems. In addition, these simulations take less than a minute on a standard PC with a Pentium II 400 MHz chip. The estimation of the parameters of these models using the method developed by Hansen and Sargent (1998) and Anderson et al. (1996) would involve the repeated solution of the algorithm as the parameters are changed to optimize a likelihood function. Thus, it is now feasible to estimate complex models of optimal experimentation.

7. Conclusion

This paper has developed a procedure for approximating optimal experimentation problems for the class of augmented LR problems. This procedure uses the perturbation method of Gaspar and Judd (1997), Judd (1998), and Jin and Judd (2002). The optimal learning problem within the context of the LR problem is modified by introducing parameters into the conditional variance–covariance matrix. These parameters introduce the possibility that either the control variables or the exogenous state vector can influence this variance–covariance matrix. This parameterization of the optimal experimentation problem includes all the examples seen in the macroeconomic literature such as Wieland (2000b, 2002, 2006), as well as more complex problems. When these parameters are zero, the optimal experimentation problem reduces to the optimal learning problem which has a well-defined solution. Thus, the perturbation procedure can be used to find the first order approximation of the optimal decision of the agents and the second order approximation of the value function.

The optimal decision under experimentation, (8), is a linear combination of the usual solution found by iterating on the Riccati equation and a term that captures the effect of uncertainty on the value function of the agent, (11). This second term uses four matrices as inputs which consist of derivatives of the equations of motion for the state vector and the conditional variance–covariance matrix from the Kalman Filter. The formula's for these matrices are provided in Appendix A. As a result optimal experimentation can be analyzed for any augmented LR problem. To implement this program: first define the matrices for your particular problem as in

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Hansen and Sargent (1998). Second, apply the formulas given in Appendix A for the four matrices in Eqs. (11) and (12). Third, iterate on the first order difference equation, (11), to measure the impact of the conditional variance–covariance matrix on the marginal value of each state variable. The final step implements Eq. (12), which yields the effect of optimal experimentation on the optimal decision of the agent. Cosimano and Gapen (2006) implements this algorithm to approximate optimal experimentation in the neighborhood of any augmented LR problem.

Implementation of this algorithm allows for the empirical evaluation of optimal experimentation on the optimal decisions of agents. Once the optimal decision for a particular problem is known, such as the optimal loan and deposit rate decisions found in Section 6, the estimation procedure of Anderson et al. (1996) and Hansen and Sargent (1998) can be modified by replacing the LR solution with the optimal experimentation solution. The estimates of the underlying parameters are found by optimizing the likelihood function built on this algorithm. It is feasible to estimate the effect of experimentation since each iteration on this algorithm takes less than a minute on a standard PC. Thus, the impact of optimal experimentation on optimal decisions of agents can be accomplished for a large class of applied economic problems outlined by Hansen and Sargent (1998).

Acknowledgments

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Appendix A

A.1. Derivatives of F

Let u_t have dimension $p \times 1$, y_t have dimension $q \times 1$, z_t have dimension $r \times 1$, and a_t have dimension $s \times 1$. The matrices have the dimensions so that product is well defined.

$$\frac{\partial F}{\partial u_t} = vec([B'_y, 0']) + vec\left(\left[\frac{\partial K_t a_t}{\partial u_t}, 0'\right]\right).$$ [32]

[32] See Theorem 6.3, p. 43 and item 4, p. 50 of Rogers (1980).

$K_t a_t$ is dependent on V_t. As a result, look at

$$\frac{\partial V_t}{\partial u_t} = \frac{\partial \tau_1 u_t'}{\partial u_t} V_3 u_t \tau_1' + (\tau_1 u_t' \otimes I_p)(V_3 \otimes I_p)\frac{\partial u_t \tau_1'}{\partial u_t},$$

which is zero for $\tau_1 = 0$.[33] Thus, $\partial K_t a_t / \partial u_t = 0$, so that $\partial F/\partial u_t = vec([B_y', 0'])$.

$$\frac{\partial F}{\partial \hat{y}_t} = vec([A_{yy}', 0']) + vec\left(\left[\frac{\partial K_t a_t}{\partial \hat{y}_t}, 0'\right]\right),$$

where

$$\frac{\partial \begin{pmatrix} K_t a_t \\ 0 \end{pmatrix}}{\partial \hat{y}_t} = -\left(\begin{pmatrix} K_t \\ 0 \end{pmatrix} \otimes I_q\right)(C_{sy} \otimes I_q) vec(I_q).^{34}$$

Here 0 has dimension $r \times s$. Thus,

$$\frac{\partial F}{\partial \hat{y}_t} = vec([A_{yy}', 0']) - \left(\begin{pmatrix} K_t \\ 0 \end{pmatrix} \otimes I_q\right)(C_{sy} \otimes I_q) vec(I_q),$$

$$\frac{\partial F}{\partial z_t} = vec([A_{yz}', A_{zz}']) + vec\left(\left[\frac{\partial K_t a_t}{\partial z_t}, 0'\right]\right).$$

$K_t a_t$ is dependent on z_t through V_t, as a result look at

$$\frac{\partial V_t}{\partial z_t} = \frac{\partial \tau_2 z_t'}{\partial z_t} V_4 z_t \tau_2' + (\tau_2 z_t' \otimes I_r)(V_4 \otimes I_r)\frac{\partial z_t \tau_2'}{\partial z_t},$$

which is zero for $\tau_2 = 0$. Thus, $\partial K_t a_t / \partial z_t = 0$, so that $\partial F/\partial z_t = vec([A_{yz}', A_{zz}'])$.

$$\frac{\partial F}{\partial \Sigma_t} = \begin{pmatrix} \frac{\partial K_t a_t}{\partial \Sigma_t} \\ 0 \end{pmatrix},$$

where 0 has dimension $qr \times q$. $K_t a_t$ is the product of three matrices $X \equiv A_{yy}\Sigma_t C_{sy}'$, $Y \equiv (C_{sy}\Sigma_t C_{sy}' + HH')^{-1}$ and $Z \equiv a_t$. By the product rule for differentiation of matrices[35]

$$\frac{\partial K_t a_t}{\partial \Sigma_t} = \frac{\partial X}{\partial \Sigma_t}(Y \otimes I_q)(Z \otimes I_q) + (X \otimes I_q)\frac{\partial Y}{\partial \Sigma_t}(Z \otimes I_q).$$

Next,

$$\frac{\partial X}{\partial \Sigma_t} = \frac{\partial A_{yy}\Sigma_t C_{sy}'}{\partial \Sigma_t} = vec(A_{yy}') vec(C_{sy}')',^{36}$$

[33]See Theorem 6.4 from Rogers (1980, p. 43).
[34]See item 16, Rogers (1980, p. 53).
[35]a_t is not dependent on Σ_t.
[36]See item 13, Rogers (1980, p. 52).

which has dimension $q^2 \times qs$.

$$\frac{\partial Y}{\partial \Sigma_t} = -((C_{sy}\Sigma_t C'_{sy} + HH')^{-1} \otimes I_q)vec(C'_{sy})vec(C_{sy})'((C_{sy}\Sigma_t C'_{sy} + HH')^{-1} \otimes I_q),$$ [37]

which has dimension $sq \times sq$.

$$\begin{aligned}\frac{\partial K_t a_t}{\partial \Sigma_t} &= vec(A'_{yy})vec(C'_{sy})'((C_{sy}\Sigma_t C'_{sy} + HH')^{-1} \otimes I_q)(a_t \otimes I_q) \\ &\quad - (A_{yy}\Sigma_t C'_{sy} \otimes I_q)((C_{sy}\Sigma_t C'_{sy} + HH')^{-1} \otimes I_q) \\ &\quad \times vec(C'_{sy})vec(C_{sy})'((C_{sy}\Sigma_t C'_{sy} + HH')^{-1} \otimes I_q)(a_t \otimes I_q).\end{aligned}$$

Next look at the effect of the perturbation vector

$$\frac{\partial F}{\partial \tau_1} = \begin{pmatrix} \frac{\partial K_t a_t}{\partial \tau_1} \\ 0 \end{pmatrix}.$$

$K_t a_t$ is dependent on τ_1 through V_t, as a result look at

$$\frac{\partial V_t}{\partial \tau_1} = \frac{\partial \tau_1 u'_t}{\partial \tau_1} V_3 u_t \tau'_1 + (\tau_1 u'_t \otimes I_p)(V_3 \otimes I_p)\frac{\partial u_t \tau'_1}{\partial u_t},$$

which is zero for $\tau_1 = 0$. Thus, $\partial K_t a_t / \partial \tau_1 = 0$, so that $\partial F / \partial \tau_1 = 0$. It also follows immediately that $\partial F / \partial \tau_2 = 0$.

Now turn to the second order derivatives.

$$\frac{\partial^2 F}{\partial u_t \partial \Sigma_t} = \begin{pmatrix} \frac{\partial^2 K_t a_t}{\partial u_t \partial \Sigma_t} \\ 0 \end{pmatrix}.$$

$\partial V_t / \partial u_t$ is independent of Σ_t so that all the terms in this second order derivatives are dependent on this derivative. Thus, $\partial^2 F / \partial u_t \partial \Sigma_t = 0$.

$$\begin{aligned}\frac{\partial^2 F}{\partial y_t \partial \Sigma_t} &= -\frac{\partial \begin{pmatrix} K_t \otimes I_q \\ 0 \otimes I_q \end{pmatrix}}{\partial \Sigma_t}[(C_{sy} \otimes I_q)Vec(I_q) \otimes I_s] \\ &= -\begin{pmatrix} (I_{(q,q)} \otimes I_q)\left(I_q \otimes \frac{\partial K_t}{\partial \Sigma_t}\right)(I_{(s,q)} \otimes I_q) \\ 0 \end{pmatrix}[(C_{sy} \otimes I_q)Vec(I_q) \otimes I_q],\end{aligned}$$ [38]

where $I_{(s,q)}$ is the commutation matrix and 0 has dimension $q^2 r \times q^2$. The partial derivative is

$$\begin{aligned}\frac{\partial K_t}{\partial \Sigma_t} &= [vec(A'_{yy}) - (K_t \otimes I_q)vec(C'_{sy})]vec(C_{sy})'((C_{sy}\Sigma_t C'_{sy} + HH')^{-1} \otimes I_q) \\ &= [vec(A'_{yy}) - vec(C_{sy}K'_t)]vec(C_{sy})'((C_{sy}\Sigma_t C'_{sy} + HH')^{-1} \otimes I_q).\end{aligned}$$ [39]

[37] See item 4, Rogers (1980, p. 50).

[38] See Theorem 6.6 of Rogers (1980, p. 45).

[39] By Theorem 4.1 of Rogers (1980, p. 21), $vec(XYZ) = (Z' \otimes X)vec(Y)$.

The final second order derivative is

$$\frac{\partial^2 F}{\partial \Sigma_t^2} = \begin{pmatrix} \frac{\partial^2 K_t a_t}{\partial \Sigma_t^2} \\ 0 \end{pmatrix},$$

where 0 has dimension $q^2 r \times q^2$.

Let $S \equiv (I_{(q,s)} \otimes I_q)(I_q \otimes \partial(C_{sy}\Sigma_t C'_{sy} + HH')^{-1}/\partial\Sigma_t)(I_{(s,q)} \otimes I_q)$ and $T \equiv (I_{(q,q)} \otimes I_q) \times (I_q \otimes vec(A'_{yy})vec(C'_{sy})')(I_{(s,q)} \otimes I_q)$ so that

$$\begin{aligned}\frac{\partial^2 K_t a_t}{\partial \Sigma_t^2} = {} & [vec(A'_{yy})vec(C'_{sy})' \otimes I_q]S[(a_t \otimes I_q) \otimes I_q] \\ & - T[((C_{sy}\Sigma_t C'_{sy} + HH')^{-1} \otimes I_q) \otimes I_q] \\ & \times[(vec(C'_{sy})vec(C'_{sy})'((C_{sy}\Sigma_t C'_{sy} + HH')^{-1} a_t \otimes I_q)) \otimes I_q] \\ & - [(A_{yy}\Sigma_t C'_{sy} \otimes I_q) \otimes I_q]S \\ & \times[(vec(C'_{sy})vec(C'_{sy})'((C_{sy}\Sigma_t C'_{sy} + HH')^{-1} a_t \otimes I_q)) \otimes I_q] \\ & - [(A_{yy}\Sigma_t C'_{sy} \otimes I_q)((C_{sy}\Sigma_t C'_{sy} + HH')^{-1} \otimes I_q) \otimes I_q] \\ & \times[[vec(C'_{sy})vec(C'_{sy})' \otimes I_q]S[(a_t \otimes I_q) \otimes I_q]],\end{aligned}$$

where the partial derivative in S is calculated above in $\partial Y/\partial\Sigma_t$.

A.2. Derivatives of G

u_t, z_t, and τ only effect G through the Kalman Gain which in turn is influenced by V_t. As a result $\partial G/\partial u_t$, $\partial G/\partial z_t$ and $\partial G/\partial \tau$ are all zero. Next,

$$\begin{aligned}\frac{\partial G}{\partial \Sigma_t} = {} & vec(A'_{yy})vec(A'_{yy})' \\ & - vec(A'_{yy})vec(C'_{sy})'[(C_{sy}\Sigma_t C'_{sy} + HH')^{-1} \otimes I_q][C_{sy}\Sigma_t A'_{yy} \otimes I_q] \\ & + [A_{yy}\Sigma_t C'_{sy} \otimes I_q]((C_{sy}\Sigma_t C'_{sy} + HH')^{-1} \otimes I_q)vec(C'_{sy})vec(C_{sy})' \\ & \times((C_{sy}\Sigma_t C'_{sy} + HH')^{-1} \otimes I_q)[C_{sy}\Sigma_t A'_{yy} \otimes I_q] \\ & - [A_{yy}\Sigma_t C'_{sy} \otimes I_q][(C_{sy}\Sigma_t C'_{sy} + HH')^{-1} \otimes I_q]vec(C'_{sy})vec(A'_{yy})'.\end{aligned}$$

Let $U \equiv (I_{(q,s)} \otimes I_q)(I_q \otimes vec(C'_{sy})vec(A'_{yy})')(I_{(q,q)} \otimes I_q)$ so that the second order partial derivative is

$$\begin{aligned}\frac{\partial^2 G}{\partial \Sigma_t^2} = {} & -[vec(A'_{yy})vec(C'_{sy})' \otimes I_q]S[(C_{sy}\Sigma_t A'_{yy} \otimes I_q) \otimes I_q] \\ & -[(vec(A'_{yy})vec(C'_{sy})'[(C_{sy}\Sigma_t C'_{sy} + HH')^{-1} \otimes I_q]) \otimes I_q]U \\ & - T[(((C_{sy}\Sigma_t C'_{sy} + HH')^{-1} \otimes I_s)vec(C'_{sy})vec(C_{sy})'\end{aligned}$$

$$
\begin{aligned}
&\times((C_{sy}\Sigma_t C'_{sy}+HH')^{-1}\otimes I_s)[C_{sy}\Sigma_t A'_{yy}\otimes I_q])\otimes I_q]\\
&+[[A_{yy}\Sigma_t C'_{sy}\otimes I_q]\otimes I_q]\\
&\times\{S[(vec(C'_{sy})vec(C_{sy})'((C_{sy}\Sigma_t C'_{sy}+HH')^{-1}\otimes I_q))\otimes I_q]\\
&+[(((C_{sy}\Sigma_t C'_{sy}+HH')^{-1}\otimes I_q)vec(C'_{sy})vec(C_{sy})')\otimes I_q]S\}\\
&\times[[C_{sy}\Sigma_t A'_{yy}\otimes I_q]\otimes I_q]\\
&+\{([A_{yy}\Sigma_t C'_{sy}\otimes I_q]((C_{sy}\Sigma_t C'_{sy}+HH')^{-1}\otimes I_q)vec(C'_{sy})vec(C_{sy})'\\
&\times((C_{sy}\Sigma_t C'_{sy}+HH')^{-1}\otimes I_q))\otimes I_q\}U\\
&-T[([(C_{sy}\Sigma_t C'_{sy}+HH')^{-1}\otimes I_q]vec(C'_{sy})vec(A'_{yy})')\otimes I_q]\\
&-[[A_{yy}\Sigma_t C'_{sy}\otimes I_q]\otimes I_q]S[vec(C'_{sy})vec(A'_{yy})'\otimes I_s].
\end{aligned}
$$

A.3. Derivation of (11)

The second order effects on the value function are calculated by taking the total differentiation of (10) with respect to the $q+r$ state variables to yield $q+r$ difference equations for each variance–covariance term

$$
\begin{aligned}
&V_{Ik}[\hat{y}_t, z_t, \Sigma_t, \tau]\\
&\quad = \mathrm{E}_t[\beta V_{jl}[\hat{y}_{t+1}, z_{t+1}, \Sigma_{t+1}, \tau](F^l_k[u_t, \hat{y}_t, z_t, \Sigma_t, \tau] + F^l_\alpha[u_t, \hat{y}_t, z_t, \Sigma_t, \tau]u^\alpha_k[\hat{y}_t, z_t, \Sigma_t, \tau])\\
&\qquad \times F^j_I[u_t, \hat{y}_t, z_t, \Sigma_t, \tau] + \beta V_{jL}[\hat{y}_{t+1}, z_{t+1}, \Sigma_{t+1}, \tau](G^{\mathrm{L}}_k[u_t, z_t, \Sigma_t, \tau]\\
&\qquad + G^{\mathrm{L}}_\alpha[u_t, z_t, \Sigma_t, \tau]u^\alpha_k[\hat{y}_t, z_t, \Sigma_t, \tau])F^j_I[u_t, \hat{y}_t, z_t, \Sigma_t, \tau]\\
&\qquad +\beta V_{Jl}[\hat{y}_{t+1}, z_{t+1}, \Sigma_{t+1}, \tau](F^l_k[u_t, \hat{y}_t, z_t, \Sigma_t, \tau] + F^l_\alpha[u_t, \hat{y}_t, z_t, \Sigma_t, \tau]u^\alpha_k[\hat{y}_t, z_t, \Sigma_t, \tau])\\
&\qquad \times G^J_I[u_t, z_t, \Sigma_t, \tau] + \beta V_{JL}[\hat{y}_{t+1}, z_{t+1}, \Sigma_{t+1}, \tau](G^{\mathrm{L}}_k[u_t, z_t, \Sigma_t, \tau]\\
&\qquad + G^{\mathrm{L}}_\alpha[u_t, z_t, \Sigma_t, \tau]u^\alpha_k[\hat{y}_t, z_t, \Sigma_t, \tau])G^J_I[u_t, z_t, \Sigma_t, \tau]\\
&\qquad + \beta V_j[\hat{y}_{t+1}, z_{t+1}, \Sigma_{t+1}, \tau](F^j_{Ik}[u_t, \hat{y}_t, z_t, \Sigma_t, \tau]\\
&\qquad + F^j_{I\alpha}[u_t, \hat{y}_t, z_t, \Sigma_t, \tau]u^\alpha_k[\hat{y}_t, z_t, \Sigma_t, \tau]) + \beta V_J[\hat{y}_{t+1}, z_{t+1}, \Sigma_{t+1}, \tau](G^J_{Ik}[u_t, z_t, \Sigma_t, \tau]\\
&\qquad + G^J_{I\alpha}[u_t, z_t, \Sigma_t, \tau]u^\alpha_k[\hat{y}_t, z_t, \Sigma_t, \tau])].
\end{aligned}
$$

If the perturbation vector is set equal to zero, then by Lemma 1 this equation becomes (11). In these calculations, use the result that $\mathrm{E}_t[F^j_I]$ is zero, since $\mathrm{E}_t[a_t]=0$.

A.4. Derivation of V_{IJ}

The second order partial derivatives of the value function with respect to the elements of the variance–covariance matrix satisfies

$$
\begin{aligned}
&V_{IJ}[\hat{y}_t, z_t, \Sigma_t, \tau]\\
&\quad = \mathrm{E}_t[\beta V_{jl}[\hat{y}_{t+1}, z_{t+1}, \Sigma_{t+1}, \tau](F^l_J[u_t, \hat{y}_t, z_t, \Sigma_t, \tau] + F^l_\alpha[u_t, \hat{y}_t, z_t, \Sigma_t, \tau]u^\alpha_J[\hat{y}_t, z_t, \Sigma_t, \tau])\\
&\qquad \times F^j_I[u_t, \hat{y}_t, z_t, \Sigma_t, \tau] + \beta V_{jL}[\hat{y}_{t+1}, z_{t+1}, \Sigma_{t+1}, \tau](G^{\mathrm{L}}_J[u_t, z_t, \Sigma_t, \tau]
\end{aligned}
$$

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$$
\begin{aligned}
&+ G^{\mathrm{L}}_{\alpha}[u_t, z_t, \Sigma_t, \tau]u^{\alpha}_J[\hat{y}_t, z_t, \Sigma_t, \tau])F^j_I[u_t, \hat{y}_t, z_t, \Sigma_t, \tau] + \beta V_j[\hat{y}_{t+1}, z_{t+1}, \Sigma_{t+1}, \tau] \\
&\times(F^j_{IJ}[u_t, \hat{y}_t, z_t, \Sigma_t, \tau] + F^j_{I\alpha}[u_t, \hat{y}_t, z_t, \Sigma_t, \tau]u^{\alpha}_J[\hat{y}_t, z_t, \Sigma_t, \tau]) \\
&+\beta V_{Kl}[\hat{y}_{t+1}, z_{t+1}, \Sigma_{t+1}, \tau](F^l_J[u_t, \hat{y}_t, z_t, \Sigma_t, \tau] + F^l_{\alpha}[u_t, \hat{y}_t, z_t, \Sigma_t, \tau]u^{\alpha}_J[\hat{y}_t, z_t, \Sigma_t, \tau]) \\
&\times G^K_I[u_t, z_t, \Sigma_t, \tau] + \beta V_{KL}[\hat{y}_{t+1}, z_{t+1}, \Sigma_{t+1}, \tau](G^{\mathrm{L}}_J[u_t, z_t, \Sigma_t, \tau] \\
&+ G^{\mathrm{L}}_{\alpha}[u_t, z_t, \Sigma_t, \tau]u^{\alpha}_J[\hat{y}_t, z_t, \Sigma_t, \tau])G^K_I[u_t, z_t, \Sigma_t, \tau] + \beta V_K[\hat{y}_{t+1}, z_{t+1}, \Sigma_{t+1}, \tau] \\
&\times(G^K_{IJ}[u_t, z_t, \Sigma_t, \tau] + G^K_{I\alpha}[u_t, z_t, \Sigma_t, \tau]u^{\alpha}_J[\hat{y}_t, z_t, \Sigma_t, \tau])].
\end{aligned}
$$

These second order partial derivatives are dependent on $u^{\alpha}_J[\hat{y}_t, z_t, \Sigma_t, \tau]$, however, it will turn out that these partial derivatives can be calculated independent of V_{IJ} when the perturbation vector is zero. Once this is complete, the V_{IJ} may be stacked into a $q^4 \times 1$ vector to yield a first order linear difference equation in $V_{IJ}[\hat{y}^{\mathrm{LR}}_t, z_t, \Sigma^{\mathrm{LR}}_t, 0]$.

A.5. Derivation of (12)

To find (12) first take the total differentiation of the Euler condition with respect to Σ_t for each control variable α

$$
\begin{aligned}
&\mathrm{E}_t[\Pi_{\alpha,\gamma}[u_t, y_t, z_t, \tau]u^{\gamma}_J[\hat{y}_t, z_t, \Sigma_t, \tau] \\
&\quad+\beta V_{ik}[\hat{y}_{t+1}, z_{t+1}, \Sigma_{t+1}, \tau]F^i_{\alpha}[u_t, \hat{y}_t, z_t, \Sigma_t, \tau](F^k_J[u_t, \hat{y}_t, z_t, \Sigma_t, \tau] + F^k_{\gamma}[u_t, \hat{y}_t, z_t, \Sigma_t, \tau] \\
&\quad\times u^{\gamma}_J[\hat{y}_t, z_t, \Sigma_t, \tau]) + \beta V_{iK}[\hat{y}_{t+1}, z_{t+1}, \Sigma_{t+1}, \tau]F^i_{\alpha}[u_t, \hat{y}_t, z_t, \Sigma_t, \tau](G^K_J[u_t, z_t, \Sigma_t, \tau] \\
&\quad+ G^K_{\gamma}[u_t, z_t, \Sigma_t, \tau]u^{\gamma}_J[\hat{y}_t, z_t, \Sigma_t, \tau]) + \beta V_i[\hat{y}_{t+1}, z_{t+1}, \Sigma_{t+1}, \tau](F^i_{\alpha J}[u_t, \hat{y}_t, z_t, \Sigma_t, \tau] \\
&\quad+ F^i_{\alpha\gamma}[u_t, \hat{y}_t, z_t, \Sigma_t, \tau]u^{\gamma}_J[\hat{y}_t, z_t, \Sigma_t, \tau]) + \beta V_{Ik}[\hat{y}_{t+1}, z_{t+1}, \Sigma_{t+1}, \tau]G^I_{\alpha}[u_t, z_t, \Sigma_t, \tau] \\
&\quad\times(F^k_J[u_t, \hat{y}_t, z_t, \Sigma_t, \tau] + F^k_{\gamma}[u_t, \hat{y}_t, z_t, \Sigma_t, \tau]u^{\gamma}_J[\hat{y}_t, z_t, \Sigma_t, \tau]) \\
&\quad+ \beta V_{IK}[\hat{y}_{t+1}, z_{t+1}, \Sigma_{t+1}, \tau]G^I_{\alpha}[u_t, z_t, \Sigma_t, \tau](G^K_J[u_t, z_t, \Sigma_t, \tau] + G^K_{\gamma}[u_t, z_t, \Sigma_t, \tau] \\
&\quad\times u^{\gamma}_J[\hat{y}_t, z_t, \Sigma_t, \tau]) + \beta V_I[\hat{y}_{t+1}, z_{t+1}, \Sigma_{t+1}, \tau](G^I_{\alpha J}[u_t, z_t, \Sigma_t, \tau] \\
&\quad+ G^I_{\alpha\gamma}[u_t, z_t, \Sigma_t, \tau]u^{\gamma}_J[\hat{y}_t, z_t, \Sigma_t, \tau])].
\end{aligned}
$$

This equation can be solved for u^{α}_J to yield

$$
\begin{aligned}
u^{\alpha}_J[\hat{y}_t, z_t, \Sigma_t, \tau] = &-\{\mathrm{E}_t(\Pi_{\alpha,\gamma}[u_t, y_t, z_t, \tau] \\
&+ \beta V_{ik}[\hat{y}_{t+1}, z_{t+1}, \Sigma_{t+1}, \tau]F^i_{\alpha}[u_t, \hat{y}_t, z_t, \Sigma_t, \tau]F^k_{\gamma}[u_t, \hat{y}_t, z_t, \Sigma_t, \tau] \\
&+ \beta V_{iK}[\hat{y}_{t+1}, z_{t+1}, \Sigma_{t+1}, \tau]F^i_{\alpha}[u_t, \hat{y}_t, z_t, \Sigma_t, \tau]G^K_{\gamma}[u_t, z_t, \Sigma_t, \tau] \\
&+ \beta V_i[\hat{y}_{t+1}, z_{t+1}, \Sigma_{t+1}, \tau]F^i_{\alpha\gamma}[u_t, \hat{y}_t, z_t, \Sigma_t, \tau] \\
&+ \beta V_{Ik}[\hat{y}_{t+1}, z_{t+1}, \Sigma_{t+1}, \tau]G^I_{\alpha}[u_t, z_t, \Sigma_t, \tau]F^k_{\gamma}[u_t, \hat{y}_t, z_t, \Sigma_t, \tau] \\
&+ \beta V_{IK}[\hat{y}_{t+1}, z_{t+1}, \Sigma_{t+1}, \tau]G^I_{\alpha}[u_t, z_t, \Sigma_t, \tau]G^K_{\gamma}[u_t, z_t, \Sigma_t, \tau] \\
&+ \beta V_I[\hat{y}_{t+1}, z_{t+1}, \Sigma_{t+1}, \tau]G^I_{\alpha\gamma}[u_t, z_t, \Sigma_t, \tau]\}^{-1} \\
&\times\{\mathrm{E}_t[\beta V_{ik}[\hat{y}_{t+1}, z_{t+1}, \Sigma_{t+1}, \tau]F^i_{\alpha}[u_t, \hat{y}_t, z_t, \Sigma_t, \tau]F^k_J[u_t, \hat{y}_t, z_t, \Sigma_t, \tau]
\end{aligned}
$$

$$
\begin{aligned}
&+ \beta V_{iK}[\hat{y}_{t+1}, z_{t+1}, \Sigma_{t+1}, \tau] F^i_\alpha[u_t, \hat{y}_t, z_t, \Sigma_t, \tau] G^K_J[u_t, z_t, \Sigma_t, \tau] \\
&+ \beta V_i[\hat{y}_{t+1}, z_{t+1}, \Sigma_{t+1}, \tau] F^i_{\alpha J}[u_t, \hat{y}_t, z_t, \Sigma_t, \tau] \\
&+ \beta V_{Ik}[\hat{y}_{t+1}, z_{t+1}, \Sigma_{t+1}, \tau] G^I_\alpha[u_t, z_t, \Sigma_t, \tau] F^k_J[u_t, \hat{y}_t, z_t, \Sigma_t, \tau] \\
&+ \beta V_{IK}[\hat{y}_{t+1}, z_{t+1}, \Sigma_{t+1}, \tau] G^I_\alpha[u_t, z_t, \Sigma_t, \tau] G^K_J[u_t, z_t, \Sigma_t, \tau] \\
&+ \beta V_I[\hat{y}_{t+1}, z_{t+1}, \Sigma_{t+1}, \tau] G^I_{\alpha J}[u_t, z_t, \Sigma_t, \tau]]\}.
\end{aligned}
$$

Equation for u^α_J in the text is found by using Lemma 1 to evaluate the partial derivatives at the LR solution. JJ

A.6. Proof of $V_{\mathscr{I}} = 0$ and $u^\gamma_{\mathscr{I}} = 0$

For each of the perturbation parameters in the vector, τ,

$$
\begin{aligned}
V_{\mathscr{I}}[\hat{y}_t, z_t, \Sigma_t, \tau] = &\mathrm{E}_t[\Pi_{\mathscr{I}}[u_t, y_t, z_t, \tau] + \beta V_j[\hat{y}_{t+1}, z_{t+1}, \Sigma_{t+1}, \tau] F^j_{\mathscr{I}}[u_t, \hat{y}_t, z_t, \Sigma_t, \tau] \\
&+ \beta V_J[\hat{y}_{t+1}, z_{t+1}, \Sigma_{t+1}, \tau] G^J_{\mathscr{I}}[u_t, z_t, \Sigma_t, \tau] + \beta V_{\mathscr{I}}[\hat{y}_{t+1}, z_{t+1}, \Sigma_{t+1}, \tau]].
\end{aligned}
$$

The partial derivatives, $F^j_{\mathscr{I}}$ and $G^J_{\mathscr{I}}$, are zero by Lemma 1 when they are evaluated at the LR solution. The partial derivatives of the variance–covariance terms HH' in the Kalman filter are zero when the perturbation parameters are zero. It follows that $u^\gamma_{\mathscr{I}}(\hat{y}^{\mathrm{LR}}_t, z_t, \Sigma^{\mathrm{LR}}_t, 0) = 0$ since it is dependent on the partial derivatives $F^j_{\mathscr{I}}$ and $G^J_{\mathscr{I}}$.

Appendix B. Supplementary data

Supplementary data associated with this article can be found in the online version at doi:10.1016/j.jedc.2007.03.009.

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